THE

MATHEMATICIAN.

EDITED BY

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The Editors of the Mathematician, on the completion of their first volume, beg to express their sincere thanks to those friends whose liberal assistance enabled them to commence the publication.

They trust that, as far as in a single volume it could be done, they will be considered to have fulfilled the promise made in their prospectus two years ago: and the dissertations on several subjects (some of which are treated by original methods, and others of them introduced for the first time into an English work), will secure the support and co-operation not only of the Editors' personal friends, but also of the friends of science in general.

ROYAL MILITARY ACADEMY,
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extent the plan of that work will be adopted in this; but, we trust, with some improvement and increased usefulness.

It is our intention to curtail, in some degree, the department of mathematical questions; for though we are fully impressed with a sense of the importance of this feature of the work, universal experience shows the difficulty of forming a sufficient number of new and good questions, where a fixed number must be made up by a given time; and the insertion of such as lead to mere petty details of calculation and deduction, suited only for the student's private exercise, tends not only to lead him into frivolous researches, but to create a false taste in science. We shall, hence, insert only such as involve some new principle, or require for their solution some new modes of investigation,—such as either lead to results remarkable for their unexpected simplicity, elegance, and symmetry; or tend to the extension of an old, or to the commencement of a new, and valuable course of inquiry. These, and these only, will find a place in the present work; and as we do not confine ourselves to any exact number of questions, or our correspondents to the time when they transmit their answers (leaving this to their own convenience), we hope to render this department free from the reproach so often applied to works of this class—that of "creating a race of mere problem-solvers."

Another feature of our work will consist in the attempt to simplify both the investigations and modes of actual operation of various elementary processes, and to furnish good models for the young student's imitation in conducting his own researches, and in putting them into a finished form. Moreover, by reducing to simple and symmetrical processes several of the operations of arithmetic and algebra, to diminish the inevitable labour of calculation, which the more experienced mathematician is often compelle to undertake in the prosecution of his more recondite researches. Amongs these, Mr. Horner's paper on *Equations and Differences* (being a co tinuation of a former paper, published in the Philosophical Transacti and reprinted in the Lady's Diary for 1838,) read before the Royal Soc in 1823, will be printed from the original manuscript—not a single ex from it having yet been published.

A third object will be to develop in sufficient detail those *in geometry*, which have been devised and carried out to consi extent by the Continental Mathematicians. The spirit of what emphatically termed *the Modern Geometry*, as well as its extr results, will be adequately illustrated. The *Descriptive Ge* *Method of Poles*, the *Geometry of the Rule*, the *Method of* the *Anharmonic Ratio*, and *Transversals*:—in short, what

to *method* in treating the geometry of two and three dimensions, and the geometry of spherical co-ordinates, with original speculations in most of them, will be more or less discussed in future numbers of this work.

In the next place, with respect to the branches of science which may be comprised, without violation of ordinary language, under one general term, the *transcendent calculus*, it is to be understood that whatever tends to the improvement of general processes, or the solution of important individual difficulties, will always receive particular attention and encouragement. The numberless papers, however, which are scattered through different academical collections, and the equally numberless works devoted to physical science, in which such subjects are treated, render it difficult for young mathematicians to discover whether they have been anticipated or not, and often almost equally difficult to ourselves. We shall, hence, generally lay down as a principle, that any real improvement upon the processes to be found in the most modern English, and the most generally accessible French works, is suitable for our pages, whether anticipated by other inquirers or not.

The application of mathematics to physical science and to the arts of life, subjected only to the condition above stated, will always be welcomed by us, and receive due attention. Moreover, elementary illustrations of the manner of using general formulæ in special physical researches, will often be given, for the purpose of enabling the young student to comprehend more clearly the force and meaning of those formulæ, and to exercise his own powers more safely in their use. We shall, likewise, give, with much satisfaction, such improvements in the application of mathematics to the demands of the military and civil engineer, the architect, the navigator, and the practical astronomer, as shall either be discovered by ourselves or furnished by our friends and correspondents.

In a commercial point of view, such undertakings as the present, have invariably been attended with considerable loss, and we have no reason to anticipate that in the present case the result would, under ordinary circumstances, be materially different. The Editors will gladly give the requisite time and labour to the management of the work; but they cannot be expected to make more than a moderate pecuniary sacrifice. This, our only difficulty, has, however, been thus obviated:—A few of our personal friends had formed themselves into a society, for raising a small annual fund to meet that part of the expenses of the publication, which should not be covered by the returns from its sale; and several others have since joined us. The funds, however, are still such as to prevent our publishing so often as we could desire; but we are not without a hope that a sufficient

number of supporters will yet come forward to enable us to accomplish a purpose, the advantages of which to mathematical science are sufficiently obvious.

The subscription is one pound per annum, paid in advance (by a post-order in favour of Mr. Rutherford, who has consented to take the pecuniary management); and each member will be entitled to one copy of the work, thus reducing his actual sacrifice to a very small sum per annum. Even this, we hope, by an increase of members, will be afterwards reduced still further; and a statement of the funds will be annually furnished to each member of the society.

In conclusion, as in this undertaking we do not consult our own personal advantage, but are actuated solely by a desire to be *really useful* to the young and ardent student, we trust that our effort will meet with the encouragement which we feel that our *motive*, leaving ability out of the question, really merits.

Royal Military Academy,
September, 1843.

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ON CONDITIONAL COEFFICIENTS; AND ON SOME ELEMENTARY EXPANSIONS.

[*Mr. Davies.*]

The ordinary phrase *indeterminate coefficients*, has been so often remarked, by accurate writers, to be not only inappropriate, but incorrect, that no apology can be required for the introduction of a substitute for it, which shall distinctly express the idea attached to this class of functions.

The phrase, *coefficients of indeterminate quantities*, might, perhaps, be unobjectionable on any ground besides its length: and many able writers have proposed to substitute other terms, — as *assumed*, *expansional*, *arbitrary*, *unknown*, and *undetermined coefficients*. To all these some objection at once occurs: for the functions are not, in the strict sense of the word, assumed, but symbolised; they come into use in other inquiries besides those which relate to expansions; they are, though, *a priori*, unknown, subject to known and given conditions; they are not arbitrary, since at the very outset they are deprived of all that can be so called, by the objects they are required to fulfil; and, finally, though they are, in their original introduction, undetermined as to value, they are capable of determination, and their values are required to be actually found.

The coefficients now under consideration, are always introduced into analysis, subjected to a series of *specified conditions;* and, as they never appear unaccompanied by these conditions, the term here chosen appears to be a very appropriate one. It is, therefore, adopted in the present and some succeeding papers in this work. This term has also the advantage of distinct and expressive application to another class of functions, nearly allied in its philosophical principles to this—known by the name of *indeterminate multipliers*, or *arbitrary multipliers*. These multipliers are only introduced conditionally; that is, the several products are subjected to predetermined combinations, or made to fulfil certain specified conditions. It is, therefore, proposed to call these *conditional multipliers*.

The ordinary method of establishing the fundamental proposition respecting conditional coefficients, has often been objected to; and with good reason too, whether the following views shall be found correct, or not. It is, in fact, one of the most striking instances of what Berkeley terms "shifting the hypothesis;" that is, making a new hypothesis for the purposes of reasoning, which is inconsistent with the existence of the original one, upon which the proposition is founded. The method of making x alternately zero and arbitrary is, however, the only one used in almost every work, even the most modern, that has been written for academical purposes, either in England or France. About eight years ago, however,

I proposed a different method of proceeding in respect of these reasonings, in Dr. Gregory's 11th edition of *Hutton's Course*, vol. i. page 239; and it turned upon the following propositions.

1. "No determinate equation of the n^{th} degree can have more than n different values of x."

2. "If an equation of the n^{th} degree can be fulfilled by more than n values of x, it will be fulfilled by *all* values of x."

3. "In case the equation can be fulfilled by more than n values of x, it is fulfilled by the coefficients of the different powers of x being contemporaneously zero."

This argument applies to all equations whose number of terms is finite; and it likewise meets precisely the problem of *partial fractions*, and the method of treating *porisms* algebraically. To the very important and constantly required subject of expansions it does not (as I had then overlooked) apply directly and satisfactorily: though minds accustomed to ulterior researches, might perhaps consider the following argument valid.

"Whatever is true *up to* the limit is true *at* the limit:" but for any finite number of terms, however great, it is true on the demonstrated principle: hence, since it is true for any number of terms *limited by infinity*, it is also true for an infinite series of terms." Whether some minds will not find the *hiatus* too extended for the admission of the conclusion, or not, is, however, when taken in connection with the following paper, a matter of little importance.

Some time ago, I was led to consider the subject of elementary expansion by means of conditional coefficients, under a different, and, it is believed, a new aspect. This applies to either a finite or an infinite series of terms with equal facility; and the first statement given of it was in my recent edition (the 12th) of vol. ii. *of Hutton's Course*, page 354. The principle there enunciated will be here amplified, in connection with some collateral remarks upon the expansions usually deduced by means of this instrument of development.*

Throughout our algebra, even from its commencement, we have to employ symbols of operation as well as of quantity. All our algebraic expressions are composed of *directed operations;* and all our algebraic equations are mere statements of the symbolical results arising out of the conditions of the problem proposed for discussion.

In some cases we may perform the prescribed operations in their prescribed order, and obtain our symbolical results without any special contrivance whatever: in others, it is necessary to rest upon proofs already offered, that the order of certain operations may be changed without affecting the accuracy of the results, in order to simplify the expression as much as possible: whilst, again, in others, (and these the most numerous and important) an entirely new series of operations must be substituted for the directed ones, to so far diminish the actual labor as to bring it within the reach of human perseverance to perform.

For this last purpose, we have recourse, almost invariably, to the method

* It requires us but to turn to the 12th ed. of Hutton, vol. i., pages 128 and 524, and to the Lady's Diary, for 1838, to be convinced of the superiority in point of clearness of the proof of *Synthetic Division*, by the process here discussed. The essential amplification of the first proof (if *proof* it can be called) referred to, when compared with the second, is sufficient to decide the question respecting my own two investigations in Hutton. Also, in comparison with Mr. Horner's own proof, given in the Diary, the clearness to the mind is much in favor of the method of conditional coefficients, the actual writing may be in favor of the original method of my gifted and

of conditional coefficients: and, indeed, from the fact of its immediate applicability to all, and of its inevitable necessity in the greater number of cases, we shall not much err if we consider it to be the true foundation of all transformation and of all development. Even where other *special* contrivances enable us to expand some classes of functions, it will be found that this method effects the same purpose with incomparably greater simplicity, brevity, and elegance: and hence, whether for the purposes of cultivating the young mind in respect of the truths already known, or of deducing for ourselves new and general theorems, it is preferable to the more recondite but isolated processes occasionally used. The strictly elementary and obvious character of the fundamental principle, and the simplicity and directness of the subsequent operations, are strong arguments in its favor.

It is seldom difficult to discover the *general form* according to which the indices of the powers of the letter or letters, according to which the expansion is to be effected, should be arranged in succession. However, whether difficult or not in any given case, this previous knowledge is essential, not only to this particular method, but to all methods whatever, as a preliminary step. A few experiments will, at all events, usually shew the *form* of the series: and it may be generally assumed, that if these letters have no fractional indices in the original expression, they will have none in the expansion; and that if they have fractional indices in the original expression, there will be no fractional ones in the development, except integer multiples of those indices,* or fractions differing from them by integers.

Under these circumstances, then, we *invariably prescribe the form* of the expansion, whether conditional coefficients or other methods be employed for the determination of the quantities independent of the letters of expansion; and the particular order so chosen is that of regularly ascending or descending integer indices. The equation is merely the statement that its second side is the actual result of the performance of the operations indicated on the left; in other words, that the two sides are *identical* in every thing but *form*. The values of the conditional coefficients are yet unknown, and the object of the process is to determine those values, and, if possible, the manner in which any one is dependent upon one or more of the preceding ones. This taken as the basis of our operations, the identity of the two sides of the equation

$$f(x) = A + Bx + Cx^2 + Dx^3 + \ldots\ldots$$

$$\text{or, } f(x) = A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \ \ldots\ldots\ldots$$

and the coefficients A, B, C, D, *etc.*, are subjected to fulfil the conditions implied in these equalities. If other forms of development should be required, the process is still the same.

The identity of the two sides in respect of value, then leads to this further equation (we shall exemplify by the former of those given above);

$$F f(x) = F \{ A + Bx + Cx^2 + Dx^3 + \ldots . \} \ldots\ldots\ldots (1)$$

* Of this general principle, though many attempts have been made, no unobjectionable proof has yet been given. It is not our present object, nor does it conduce to any important elementary object, to discover whether a function can be developed in indefinitely numerous forms, or only in a given specific number. In all cases we merely allege that the function *can be* developed in certain specified forms; whilst it is neither affirmed nor denied that it can be developed in *no other*.

in which, again, *the expanded results must be identical: and identity can only exist, the symbol x being indeterminate, by the coefficients of the like powers of x, on both sides being identical in value.* Instead, therefore, of a formal proof of this equality, we see that it is only the statement of the condition of the second side being the expansion of the first. Any attempt, therefore, when the principle is fairly stated, to give a formal proof is hence both illogical and superfluous. The only change we have made is, indeed, but the change of statement into the form of a problem from that of a theorem, as usually given; and the consequence is, that we are now required to find those values of the coefficients of both expressions in (1) to produce an identity,—this identity, implying the equality of the coefficients of the homologous terms on both sides, and hence giving the several equations (as many in number as there are coefficients, A, B, C, *etc.*) from which the unknown coefficients can be found. The values thus found must render the two sides identical; and, hence, when brought to one side, all the coefficients will be zero.

The possibility of the expansion in the assigned form will be decided in all cases by the values of A, B, C, *etc.*, deduced from the equality of the homologous coefficients being real; and the problem is at the same time resolved. Should, on the other hand, the problem be impossible, we shall discover this by some contradiction amongst the same set of equations; such as imaginary, ambiguous, or infinite values, or some contradiction to pre-determined principles. The latter circumstance can only arise from our neglect of some fact which was essential in the investigation, or is, in reality, the consequence of some fundamental mistake in our application of the method to the case in which that contradiction appears. The former might occur in every shape specified; but it most usually does so in the form of imaginary values of the coefficients. When any coefficient is 0, the power, whose coefficient it is, does not appear in the development; when one coefficient is infinite, others involving this (and some of them always do) will be infinite also; and the difference of two infinites is always some finite but indeterminate quantity; and when the value of a coefficient takes the usual ambiguous form, $\frac{0}{0}$, the value is always multiplex, and the investigation of these values requires the mode of treatment usually given to the corresponding case in Taylor's theorem. In ordinary algebraic developments, however, the occurrence of infinite and ambiguous coefficients is so very rare, that I do not recollect to have met with a single instance; and this may justly create a doubt whether it can ever occur in such cases as we are considering.

The particular function F is altogether arbitrary, and is always selected according to the purposes in view: but these are best consulted by giving some kind of relation between F and f. In mere development of a numerical function, F and f are most advantageously taken the inverse of each other: and even in the deduction of general formulæ, this is particularly so, in a great number of cases.

In all operations whose mere object is transformation, no account is, or can be taken, as far as general process is concerned, of the relative values of the symbols employed. In many discussions, however, this may become an object of essential inquiry, and especially so in all matters relating to series and expansions. The most usual is, when numerical applications are to be made, and the convergency of the successive terms towards a minimum

limit, or their divergency from a given limit, becomes essential to the investigation:—the minimum limit, for most practical purposes, being *zero*. Many distinguished analysts have said that diverging series are "false;" forgetting that there is a wide difference between falsehood and inapplicability. The expansion itself which gave that series may be entirely correct, and yet the use which we may, under some circumstances, wish to make of it, not be possible. It would, indeed, be taking a very confined view of any general method, were we to consider its application to mere individual numerical purposes. A much more important and a much more frequent use to which the method of conditional coefficients is applied, is, that of investigating *general forms* for the expansion of particular and often-occurring classes of functions; as, for instance, the binomial, multinomial, exponential, logarithmic, trigonometrical, or Taylor's series. This method is, in fact, a *general instrument*, by which the expansion of any individual function can be obtained, and whereby literal formulæ are deduced for the most general of those classes of functions. Its great value, indeed, is the power which it confers of investigating these formulæ of expansion, which would otherwise require separate and isolated methods, adapted to each peculiar case. Arithmetical approximation is a subordinate object; and, indeed, except as an exercise for young students, this method is rarely resorted to for that purpose.

A recent author, whose writings display much original and profound thinking, has made a distinction between convergent and divergent series, which he calls "perfect" and "imperfect." He states that "the principle of indeterminate coefficients is true for a perfect series, but not for an imperfect one." He instances the development in series of the function $\frac{1}{1-x}$ into $1 + x + x^2 \ldots\ldots x^n + \frac{x^n+1}{1-x}$; and urges, that when x is greater than 1, this series is false. This is so manifest a confusion between the truth of a method of literal development and the specific numerical application, that it is singular that a mind so generally logical should have made it. In general, it is true, that whether in the direct development of an individual expression, or in the deduction of a general formula for that class of expressions, we satisfy ourselves with merely obtaining the relation of each coefficient to the preceding ones (or, as it is often termed, "the scale of relation" of those coefficients): but it does not thence follow, that the value of all the terms (or the "remainder") might not be also obtained, if we were to make it the object of our search, by this method, as well as by the method of division itself. It is no argument that this is not usually done, and therefore cannot be done: all that can be said is, that we have not made this an object, and hence have not been in the habit of attempting it. Our general objects in the use of the method have been those which have not required it. We may remark, moreover, that this case has no analogy to any other class of functions: from none else than those which arise from division can the "remainder" be accurately found; although in several others we can assign limits between which it must lie. However, if we choose to seek the expression for the remainder by this method, we may obtain it with as much simplicity, though not with so little actual trouble, as by common algebraic division. For, denote the expansion, stopping at the $(m+1)^{\text{th}}$ term, by

$$\frac{1}{1-x} = A + Bx + Cx^2 + \ldots\ldots + Lx^k + Mx^l + \frac{N x^m}{1-x};$$

then multiplying out, and ranging the terms in x, we get

$$\begin{array}{rl} 1 = A + Bx + Cx^2 + \ldots\ldots + Lx^k + Mx^l & \Big| + \dfrac{N x^m}{1-x} \\ - Ax - Bx^2 - \ldots\ldots - Kx^k - Lx^l & \Big| - Mx^m - \dfrac{Nx^{m+1}}{1-x}. \end{array}$$

Now, in the ordinary way, we get for the coefficients which lie to the left of the vertical line,

$$1 = A = B = C = \ldots\ldots = M;$$

and then there remain for the completion of the value of the series, the three terms to the right of that vertical line. These give, after reduction to a common denominator, and equating the coefficients to zero, to produce identity with those on the left,

$$Nx^m (1 - x) = Mx^m (1 - x), \text{ or } M = N.$$

This makes the expansion, stopping with the $(m-1)^{\text{th}}$ term as a general correction, the same that is obtained by actual division. Even, therefore, looking to the process in respect of mere calculation, there is no default in its operation.

In all cases where the function is symmetrical with respect to two quantities, it may be developed in ascending powers of either of them and descending powers of the other; and this will always furnish two separate developments of that function, each adapted to calculation when the other is not,—the one giving a convergent series when the other gives a divergent one. Let us, for instance, in the present case, take the more general form $\dfrac{1}{a-x}$; then, according as we take x in ascending or descending powers, we shall get

$$\frac{\frac{1}{a}}{1-\frac{x}{a}} = \frac{1}{a}\left\{1 + \frac{x}{a} + \frac{x^2}{a^2} + \ldots\right\} \text{ or}$$

$$-\frac{\frac{1}{x}}{1-\frac{a}{x}} = -\frac{1}{x}\left\{1 + \frac{a}{x} + \frac{a^2}{x^2} + \ldots\right\}$$

Of course when in the development we employ descending powers of x, the assumption must be in the general form, omitting the monomial factor $-\dfrac{1}{x}$,

$$\frac{1}{1-\frac{a}{x}} = A + \frac{B}{x} + \frac{C}{x^2} + \ldots\ldots$$

It will thus be seen that when x is supposed greater than a, we must, to obtain a result adapted to numerical use, employ descending powers; and when it is less than a, ascending powers of x. Each is inapplicable to the purpose where the other is applicable; and the final correction in the case of x greater than a, is of the same general form in $\dfrac{a}{x}$ that the other is in $\dfrac{x}{a}$. No objection to the method can, therefore, be taken on the ground specified above.

Another instance, which must often be a source of difficulty to a careful student, is the application of the binomial theorem with fractional indices. Take, for example, the expansion of $(a^2-x^2)^{\frac{1}{2}}$ by this formula, as given by all elementary writers. It is

$$(a^2-x^2)^{\frac{1}{2}}=a\left\{1-\frac{x^2}{2a^2}-\frac{x^4}{8a^4}-\frac{x^6}{16a^6}-\dots\dots\right\}$$

As far as the relation of the successive coefficients is concerned, those within the braces are undoubtedly correct; and a little attention to the first step would have given the double sign, $\pm$, to the factor a, that is, have prefixed $(1)^{\frac{1}{2}}$ to the whole series. The hypothesis of x being less than a is here *tacitly* made, since in the employment of the binomial theorem, we must ever recollect that the entire system of arrangement (employed in the investigation of that formula) requires that the development shall be according to ascending powers of the *second term*. To effect this development in ascending powers of a, we must, then, write

$$(a^2-x^2)^{\frac{1}{2}}=\left\{-(x^2-a^2)\right\}^{\frac{1}{2}}=(-1)^{\frac{1}{2}}x\,(1-\frac{a^2}{x^2})^{\frac{1}{2}}$$

$$=(-1)^{\frac{1}{2}}x\left\{1-\frac{a^2}{2x^2}-\frac{a^4}{8x^4}-\frac{a^6}{16x^6}-\dots\dots\right\}$$

We thus see that the incorrect result usually obtained, as the expansion of this case, arises from our neglecting a circumstance which was essentially involved in the method of investigation of the binomial theorem itself: that the result being a quantity whose value is infinity negative equal to a quantity which is essentially imaginary. We see, too, that the same circumstance requires the true development of the binomial theorem for fractional indices to be obtained from

$$(+1)^{\frac{m}{n}}a^m\left\{1+\frac{x^n}{a^n}\right\}^{\frac{m}{n}},\text{ or }(+1)^{\frac{m}{n}}x^m\left\{1+\frac{a^n}{x^n}\right\}^{\frac{m}{n}}$$

when both signs are positive; and when the second sign is negative from

$$(+1)^{\frac{m}{n}}a^m\left\{1-\frac{x^n}{a^n}\right\}^{\frac{m}{n}},\text{ or }(-1)^{\frac{m}{n}}x^m\left\{1-\frac{a^n}{x^n}\right\}^{\frac{m}{n}}$$

according as x is less or greater than a.*

Deficiencies of the same kind exist in the usual formulæ for the exponential and logarithmic series, as well as in some others: but we shall only notice those we have actually specified, on this occasion, deferring the others for a future article.

* Considering that this work will fall into the hands of mathematical teachers, to whom any hint, however trivial apparently, is valuable, if the result of experience, I would beg to make one remark on the successive steps of the operation in applying the binomial theorem to fractional indices, whether positive or negative.

It will conduce to the student's accuracy of work, by diminishing the number of cotemporaneous operations, to always transform the expansion in the manner in the text, whether the signs be both $+$ or not. He will thus have no fractional indices to attend to in the general series. The mode of putting down the numerical work for finding the coefficients given at *Page* 245, *vol.* 1., 12*th ed.*, *Hutton*, will then be advantageously adopted: the coefficient of the second term being considered *plus*. Lastly, change the signs of the *odd powers* of the second term thus obtained: and the expansion will be complete.

In expanding a^x, we tacitly assume, without specifying it, that a is positive; but upon examining the case of $(-a)^x$ as expanded in the same way, we should get

$$A = -(a+1) - \tfrac{1}{2}(a+1)^2 - \tfrac{1}{3}(a+1)^3 - \ldots = -\frac{1}{0}.$$

which renders the development *real*, whilst in the case of x becoming fractional with even denominators, it must be imaginary. At the same time, there is no restriction of the values of x in the original investigation as usually conducted. Such result therefore cannot be correct. We do, in fact, omit an important step, in both cases, though the effect of it does not appear in the result when a is positive. The true development is

$$(\pm a)^x = (\pm 1)^x \left\{ 1 + \frac{Ax}{1} + \frac{A^2x^2}{1\,.\,2} + \ldots\ldots\ldots\ldots \right\}$$

It will hence depend upon the successive infinitely numerous values of x in the original expressions, whether the expansion be real or imaginary. For every value of x, a^x will have *one* real value, and $\left(\text{putting } x = \frac{u}{v}\right)$ $(v-1)$ imaginary ones: and for the case of v being an even number, there will be v imaginary values of $(-a)^x$, whilst for v odd, there will be one real negative value of $(-a)^x$, and $(v-1)$ imaginary. In short, like the binomial theorem, the values will be the ordinary series multiplied by the v roots of $+1$, and those of -1, respectively. The inconsistency has its origin common with many other inconsistencies in elementary algebra, in our inattention to the fundamental properties of algebraic equations.

In the last place, the logarithmic expansion is also incomplete. We shall, here, for simplicity, only take the Naperian logarithms in our formulæ, as the reasoning is precisely the same, whatever the base of the system may be. That series for x in the equation $a^x = N$, is

$$x = (N-1) - \tfrac{1}{2}(N-1)^2 + \tfrac{1}{3}(N-1)^3 - \ldots\ldots$$

and when this is applied to $a^x = -N$, the form becomes

$$x = -(N+1) - \tfrac{1}{2}(N+1)^2 - \tfrac{1}{3}(N+1)^3 - \ldots\ldots$$

But we know that in this case x is imaginary,* whilst the value here given of it is infinity negative. This form, indeed, might have been deduced *à priori* in the same way as that for positive N: but its inconsistency proves its inaccuracy. Its true value is thus obtained:—

Since $e^x = (-N)$, we have $e = (-N)^{\frac{1}{x}} = (-1)^{\frac{1}{x}}\, N^{\frac{1}{x}}$; whence taking $\log_e$ of both sides, we get after reduction,

$$x = \log_e N + \log_e(-1);$$

which is essentially imaginary, since $\log_e(-1)$ is so.

For the purpose of analogy, we may write the value of x for positive N, as follows:—

$$x = \log_e N + \log_e(+1).$$

* The following is a very simple proof of this theorem.

Since $a^x = -N$, we have $a = (-N)^{\frac{1}{x}}$: but no root of a negative quantity can be positive; and hence this equation cannot be fulfilled by any real numbers.

The particular *form* given to this imaginary value, which depends upon circular functions, cannot be deduced with propriety in this stage of a systematic course of inquiry: but that will be made the subject of discussion in a future paper.

The value of x in the equation $(-a^x) = -$ N, may also be readily found: for we have by the preceding

$$x = \frac{\log_e(-\mathrm{N})}{\log_e(-a)} = \frac{\log_e \mathrm{N} + \log_e(-1)}{\log_e a + \log_e(-1)}$$

which is evidently always imaginary, except when $\mathrm{N} = a$, and then $x = 1$.

In conclusion, however, it is necessary to state, that *indeterminateness* is only one particular aspect under which a general principle in analysis appears,—that of *congruity*. It is this, that wherever congruous results are required to take place under all circumstances of an expression, they can generally be obtained only by means of the coefficients of incongruous terms of the equation being separately and simultaneously made zero. In the view already discussed, the different powers of the expansional symbols are incongruous; and the congruity can only be restored by the coefficients. The method usually employed for investigating the root of a binomial surd, is another instance: viz. that if $a + \sqrt{b} = x + \sqrt{y}$, the equation can only be fulfilled by $x = a$, $y = b$; since $a - x = \sqrt{y} - \sqrt{b}$ can be fulfilled on no other hypothesis, the difference of two roots being in no case a rational number, whilst a and x are by hypothesis rational.

Again, the imaginary expression $a + b\sqrt{-1} = x + y\sqrt{-1}$, requires that $x = a$ and $y = b$: for in no other way can the equation

$$\frac{a-x}{b-y} = -\sqrt{-1}$$

be fulfilled; since a, b, x, y, being, by hypothesis, real, the quotient can never be imaginary.

To pursue the views to which these and other similar cases lead, is forbidden here, by the circumscribed space which can be allowed for discussions of this kind, in the present number: but at a future time it is probable that the general principle may be followed out into some interesting and valuable results.

ON THE INTERSECTION OF TWO CURVES OF THE SECOND DEGREE.

[*Mr. Rutherford.*]

In many inquiries in the higher geometry it is requisite that the values of the co-ordinates of the points of intersection of two curves of the second degree should be determined, and since a curve of the second degree is represented by an equation involving both the first and second powers of the variable quantities, it becomes necessary that we must eliminate from two such general equations either of the variables, in order that, from the resulting equation involving only the remaining variable, we may obtain the values of the co-ordinates of the points of intersection with reference to that variable. This will be evident from the consideration, that at the points of intersection the values of the two variables are the same for both curves, and consequently at these points, and these only, the two equations of the curves exist simultaneously.

Let, then, the equations of two curves of the second degree referred to any axes, making an angle $= \alpha$, be

$$a_1y^2 + b_1xy + c_1x^2 + d_1y + e_1x + f_1 = 0 \ldots\ldots\ldots (1)$$
$$a_2y^2 + b_2xy + c_2x^2 + d_2y + e_2x + f_2 = 0 \ldots\ldots\ldots (2)$$

and let it be required to determine the values of x and y from these simultaneous equations.

These equations may be written in the following forms,

$$a_1y^2 + (b_1x + d_1)y + c_1x^2 + e_1x + f_1 = 0$$
$$a_2y^2 + (b_2x + d_2)y + c_2x^2 + e_2x + f_2 = 0\,;$$

and if we assume

$$\begin{array}{l|l} b_1x + d_1 = h_1 & c_1x^2 + e_1x + f_1 = k_1 \\ b_2x + d_2 = h_2 & c_2x^2 + e_2x + f_2 = k_2 \end{array}$$

then h_1, h_2, k_1, k_2, are all functions of x, and the preceding equations become

$$a_1y^2 + h_1y + k_1 = 0 \ldots\ldots\ldots (3)$$
$$a_2y^2 + h_2y + k_2 = 0 \ldots\ldots\ldots (4)$$

Let us also suppose that

$$a_1h_2 - h_1a_2 = p \ldots\ldots\ldots\ldots (5)$$
$$a_1k_2 - k_1a_2 = q \ldots\ldots\ldots\ldots (6)$$
$$h_1k_2 - k_1h_2 = r \ldots\ldots\ldots\ldots (7)$$

then multiplying (3) and (4) by a_2 and a_1 respectively, and taking the difference of these products, we get, by means of (5) and (6) the equation

$$py + q = 0 \ldots\ldots\ldots\ldots (8)$$

Again, multiplying (3) and (4) by k_2 and k_1, respectively, and subtracting the latter product from the former, we have, by (6) and (7), the equation

$$(qy + r)y = 0 \ldots\ldots\ldots\ldots (9)$$
$$\text{or } y = 0, \text{ and } qy + r = 0 \ldots\ldots\ldots\ldots (10)$$

Now that y may be $= 0$ in equation (8), we must have $q = 0$, and therefore by (6)

$$a_1k_2 - k_1a_2 = 0$$

which, since the values of k_1 and k_2 are in this case each $= 0$, is verified.

But taking the second of the equations (10), and multiplying (8) and (10) by q and p respectively; then, by subtracting the one product from the other, we get

$$q^2 = pr \ldots\ldots\ldots (11)$$

and by restoring the values of p, q, r, we have

$$(a_1k_2 - k_1a_2)^2 = (a_1h_2 - h_1a_2)\,(h_1k_2 - k_1h_2) \ldots\ldots (12)$$

and thus the variable quantity y has been eliminated, since h_1, h_2, k_1, k_2, are all functions of x. Restoring the values of these in terms of x, we get

$$a_1h_2 - h_1a_2 = (a_1b_2 - b_1a_2)\,x + (a_1d_2 - d_1a_2)$$
$$a_1k_2 - k_1a_2 = (a_1c_2 - c_1a_2)\,x^2 + (a_1e_2 - e_1a_2)\,x + (a_1f_2 - f_1a_2)$$
$$h_1k_2 - k_1h_2 = (b_1c_2 - c_1b_2)x^3 + \left.\begin{array}{l}(b_1e_2 - e_1b_2) \\ + (d_1c_2 - c_1d_2)\end{array}\right| x^2 + \left.\begin{array}{l}(b_1f_2 - f_1b_2) \\ + (d_1e_2 - e_1d_2)\end{array}\right| x + (d_1f_2 - f_1d_2)$$

Substituting the values of these binomials in (12), and arranging the

result according to the powers of x, we have finally the biquadratic equation

$$\begin{aligned}
&\{(a_1c_2-c_1a_2)^2+(b_1a_2-a_1b_2)(b_1c_2-c_1b_2)\}x^4\\
&\left.\begin{aligned}+2(a_1c_2-c_1a_2)(a_1e_2-e_1a_2)+(a_1d_2-d_1a_2)(c_1b_2-b_1c_2)\\+(b_1a_2-a_1b_2)(b_1e_2-e_1b_2+d_1c_2-c_1d_2)\end{aligned}\right|x^3\\
&\left.\begin{aligned}+2(a_1c_2-c_1a_2)(a_1f_2-f_1a_2)+(b_1a_2-a_1b_2)(b_1f_2-f_1b_2+d_1e_2-e_1d_2)\\+(a_1e_2-e_1a_2)^2+(d_1a_2-a_1d_2)(b_1e_2-e_1b_2+d_1c_2-c_1d_2)\end{aligned}\right|x^2\\
&\left.\begin{aligned}+2(a_1e_2-e_1a_2)(a_1f_2-f_1a_2)+(b_1a_2-a_1b_2)(d_1f_2-f_1d_2)\\+(d_1a_2-a_1d_2)(b_1f_2-f_1b_2+d_1e_2-e_1d_2)\end{aligned}\right|x\\
&+(a_1f_2-f_1a_2)^2+(a_1d_2-d_1a_2)(f_1d_2-d_1f_2)=0\,..(13)
\end{aligned}$$

As the variable quantities x and y are symmetrically involved in each of the given equations, it is obvious that, if we had eliminated x, the resulting equation in y, would have been precisely of the same form as that we have obtained for x, having simply an interchange of the co-efficients of the first and second powers of x and y. Hence to obtain the equation for the determination of the values of y, we must write in (13) the co-efficients a_1, a_2, d_1, d_2, for c_1, c_2, e_1, e_2, and *vice versa*, and this being done, we get for the determination of the values of y, the biquadratic equation

$$\begin{aligned}
&\{(a_1c_2-c_1a_2)^2+(b_1a_2-a_1b_2)(b_1c_2-c_1b_2)\}y^4\\
&\left.\begin{aligned}+2(c_1a_2-a_1c_2)(c_1d_2-d_1c_2)+(c_1e_2-e_1c_2)(a_1b_2-b_1a_2)\\+(b_1c_2-c_1b_2)(b_1d_2-d_1b_2+e_1a_2-a_1e_2)\end{aligned}\right|y^3\\
&\left.\begin{aligned}+2(c_1a_2-a_1c_2)(c_1f_2-f_1c_2)+(b_1c_2-c_1b_2)(b_1f_2-f_1b_2+e_1d_2-d_1e_2)\\+(c_1d_2-d_1c_2)^2+(e_1c_2-c_1e_2)(b_1d_2-d_1b_2+e_1a_2-a_1e_2)\end{aligned}\right|y^2\\
&\left.\begin{aligned}+2(c_1d_2-d_1c_2)(c_1f_2-f_1c_2)+(b_1c_2-c_1b_2)(e_1f_2-f_1e_2)\\+(e_1c_2-c_1e_2)(b_1f_2-f_1b_2+e_1d_2-d_1e_2)\end{aligned}\right|y\\
&+(c_1f_2-f_1c_2)^2+(c_1e_2-e_1c_2)(f_1e_2-e_1f_2)=0\,..(14)
\end{aligned}$$

Hence the co-ordinates of the four points of intersection of the two curves of the second degree are fully determined, and from equation (13) or (14) we learn that two curves of the second degree cannot cut each other in more than four points.

If the terms involving the first powers of the variables are absent, then the origin of co-ordinates is removed to the common centre of the curves, and the equations are simply

$$a_1y^2+b_1xy+c_1x^2+f_1=0\,........(15)$$
$$a_2y^2+b_2xy+c_2x^2+f_2=0\,........(16)$$

and therefore making d_1, d_2, e_1, e_2, each equal to zero in the equations (13) and (14), they reduce to the quadratic forms

$$\begin{aligned}&\{(a_1c_2-c_1a_2)^2+(b_1a_2-a_1b_2)(b_1c_2-c_1b_2)\}x^4+\{2(a_1c_2-c_1a_2)(a_1f_2-f_1a_2)\\&\qquad+(b_1a_2-a_1b_2)(b_1f_2-f_1b_2)\}x^2+(a_1f_2-f_1a_2)^2=0\,..(17)\end{aligned}$$
$$\begin{aligned}&\{(a_1c_2-c_1a_2)^2+(b_1a_2-a_1b_2)(b_1c_2-c_1b_2)\}y^4+\{2(c_1a_2-a_1c_2)(c_1f_2-f_1c_2)\\&\qquad+(b_1c_2-c_1b_2)(b_1f_2-f_1b_2)\}y^2+(c_1f_2-f_1c_2)^2=0\,..(18)\end{aligned}$$

Further, if the co-efficients a_1, b_1, c_1, are each zero, then equation (1) reduces to

$$d_1y+e_1x+f_1=0\,............(19)$$

the locus of which is no longer a curve, but a straight line. Hence, if we

make $a_1 = b_1 = c_1 = 0$, the equations (13) and (14) immediately reduce to the quadratic forms

$$\{a_2e_1^2+d_1(d_1c_2-e_1b_2)\}x^2+\{2a_2e_1f_1+d_1(d_1e_2-e_1d_2-f_1b_2)\}x$$
$$+a_2f_1^2+d_1(d_1f_2-f_1d_2)=0\,..(20)$$
$$(a_2e_1^2+d_1(d_1c_2-e_1b_2))y^2+\{2c_2d_1f_1+e_1(e_1d_2-d_1e_2-f_1b_2)\}y$$
$$+c_2f_1^2+e_1(e_1f_2-f_1e_2)=0\,..(21)$$

and from these we can find the points of intersection of a given straight line with either of the conic sections, as well as those of a right line with a circle.

In the case of the circle we must have $a_2 = c_2 = 1$, and $b_2 = 2\cos\alpha$ in equation (2), where $\alpha =$ inclination of the axes; hence (20) and (21) become

$$\{e_1^2+d_1(d_1-2e_1\cos\alpha)\}x^2+\{2e_1f_1+d_1(d_1e_2-e_1d_2-2f_1\cos\alpha)\}x$$
$$+f_1^2+d_1(d_1f_2-f_1d_2)=0\,..(22)$$
$$\{e_1^2+d_1(d_1-2e_1\cos\alpha)\}y^2+\{2d_1f_1+e_1(e_1d_2-d_1e_2-2f_1\cos\alpha)\}y$$
$$+f_1^2+e_1(e_1f_2-f_1e_2)=0\,..(23)$$

which, when the axes are rectangular, reduce to

$$(d_1^2+e_1^2)x^2+\{2e_1f_1+d_1(d_1e_2-e_1d_2)\}x+f_1^2+d_1(d_1f_2-f_1d_2)=0\,....(24)$$
$$(d_1^2+e_1^2)y^2+\{2d_1f_1+e_1(e_1d_2-d_1e_2)\}y+f_1^2+e_1(e_1f_2-f_1e_2)=0\,....(25)$$

When the origin is at the centre of the circle, we have $d_2=0$, and $e_2=0$; hence (24) and (25) further reduce to

$$(d_1^2+e_1^2)x^2+2e_1f_1x+f_1^2+f_2d_1^2=0\,....................(26)$$
$$(d_1^2+e_1^2)y^2+2d_1f_1y+f_1^2+f_2e_1^2=0\,....................(27)$$

We shall, for illustration, resolve the following example.

Ex.—Let the equation of a central curve of the second degree be

$$ay^2+bxy+cx^2+f=0\,........(1)$$

where the origin of the co-ordinates is at the centre, and let the oblique axes be inclined at an angle $=\alpha$; then the equation of a circle referred to the same origin and axes will be

$$y^2+2xy\cos\alpha+x^2-r^2=0\,......(2)$$

Now to determine the co-ordinates of the points of intersection of the circle with the curve expressed by (1), we must employ equations (17) and (18). Comparing the coefficients of (1) and (2) with those of (15) and (16) we get

$$a_1=a,\ b_1=b,\ c_1=c,\ f_1=f$$
$$a_2=1,\ b_2=2\cos\alpha,\ c_2=1,\ f_2=-r^2;$$

therefore, by substituting these values of the coefficients of the variables in (17) and (18), we find the following resulting equations: viz.

$$\{(a-c)^2+(b-2a\cos\alpha)(b-2c\cos\alpha)\}x^4$$
$$-\{2(a-c)(ar^2+f)+(b-2a\cos\alpha)(br^2+2f\cos\alpha)\}x^2+(ar^2+f)^2=0\,..(3)$$
$$\{(a-c)^2+(b-2a\cos\alpha)(b-2c\cos\alpha)\}y^4$$
$$+\{2(a-c)(cr^2+f)-(b-2c\cos\alpha)(br^2+2f\cos\alpha)\}y^2+(cr^2+f)^2=0\,..(4)$$

The coefficient of x^4 or y^4 may be put in other forms as well as those of x^2 and y^2. For, multiplying out, and writing $1-\sin^2\alpha$ for $\cos^2\alpha$, we get

$$(a-c)^2+(b-2a\cos\alpha)(b-2c\cos\alpha)=(a-c)^2+b^2-2b(a+c)\cos\alpha+4ac\cos^2\alpha$$
$$=(a-\cos\alpha+c)^2+(b^2-4ac)\sin^2\alpha,$$

and the coefficients of x^2 and y^2 are readily transformed to

$$-r^2\{2a(a-b\cos\alpha+c)+b^2-4ac\}+2f(a-b\cos\alpha+c-2a\sin^2\alpha)$$

and, $-r^2\{2c(a-b\cos\alpha+c)+b^2-4ac\}+2f(a-b\cos\alpha+c-2c\sin^2\alpha)$.

Now, adopting the convenient notation which Mr. Davies has employed at pp. 299 and 308 of his second volume of *Hutton's Course of Mathematics*, recently published, viz.

$$Q=a-b\cos\alpha+c$$
$$R^2=(a-b\cos\alpha+c)^2+(b^2-4ac)\sin^2\alpha$$
$$H=(a-b\cos\alpha+c)-2a\sin^2\alpha=Q-2a\sin^2\alpha$$
$$K=(a-b\cos\alpha+c)-2c\sin^2\alpha=Q-2c\sin^2\alpha,$$

and substituting these in the transformed coefficients of the variables in (3) and (4), we shall have these equations changed into the following

$$R^2x^4-\{r^2(2aQ+b^2-4ac)-2fH\}x^2+(ar^2+f)^2=0\ldots.(5)$$
$$R^2y^4-\{r^2(2cQ+b^2-4ac)-2fK\}y^2+(cr^2+f)^2=0\ldots.(6)$$

and resolving either these, or equations (3) and (4), we finally obtain

$$x=\pm\left\{\frac{2(a-c)(ar^2+f)+(b-2a\cos\alpha)\{br^2+2f\cos\alpha\pm\sqrt{(br^2+2f\cos\alpha)^2-4(ar^2+f)(cr^2+f)}\}}{2R^2}\right\}^{\frac{1}{2}}\ldots\ldots..(7)$$

$$y=\pm\left\{\frac{2(c-a)(cr^2+f)+(b-2c\cos\alpha)\{br^2+2f\cos\alpha\pm\sqrt{(br^2+2f\cos\alpha)^2-4(ar^2+f)(cr^2+f)}\}}{2R^2}\right\}^{\frac{1}{2}}\ldots\ldots..(8)$$

which determine the co-ordinates of the four points of intersection of the circle with the ellipse or hyperbola.

In consequence of the symmetrical nature of the central curves with reference to their conjugate axes, they will be intersected by the circle in four points, which, two and two, range in the same straight line with the centre of the curve; and therefore the lines joining the points of intersection, two and two, must necessarily be diameters of the curve. If the expression under the radicals in the values of x and y be equal to zero, then the circle only touches the curve in two points, and the line joining these points evidently becomes a *principal* diameter. If the expression under the radicals be negative, then the circle neither cuts nor touches the curve, the value of r being too great or too small either for intersection or contact.

It may be well to remind the student that the method here employed for determining the co-ordinates of the points of intersection of two curves of the second degree, is equally applicable in the resolution of two simultaneous equations involving both the first and second powers of two unknown quantities. As an exercise the student might find all the real values of x and y from the equations

$$2y^2+5xy-3x^2-6y+4x-28=0$$
$$7y^2-6xy+2x^2+3y-3x-20=0$$

PROPERTIES OF THE "ASSOCIATED SPHERICAL TRIANGLES."

[*Mr. Fenwick.*]

In the "Lady's and Gentleman's Diary," for the present year, I proposed for demonstration, the following relation amongst the radii of the associated triangles: viz.

$$\cot^2 r + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3 = \tan^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3;$$

where, r, r_1, r_2, r_3, and R, R_1, R_2, R_3, are the radii of the inscribed and circumscribed circles. This relation is the more remarkable, inasmuch as the equality $\cot r + \cot r_1 + \cot r_2 + \cot r_3 = \tan R + \tan R_1 + \tan R_2 + \tan R_3$, is also known to exist.

Those properties that follow, in which

$$n^2 = \sin s \sin(s-a) \sin(s-b) \sin(s-c),$$
$$N^2 = -\cos S \cos(S-A) \cos(S-B) \cos(S-C),$$

are equally remarkable for their elegance and symmetry. It would be easy to extend the series of properties, as the combinations of the radii seem inexhaustible, but enough is done, it is hoped, to direct attention to this interesting class of inquiries.

We shall first employ the following values of $\tan r$, $\tan r_1$, &c., given by De Gua: viz.

$$\tan r = \frac{n}{\sin s}, \quad \tan r_1 = \frac{n}{\sin(s-a)},$$
$$\tan r_2 = \frac{n}{\sin(s-b)}, \quad \tan r_3 = \frac{n}{\sin(s-c)}.$$

By multiplication, we get

$$\tan r \tan r_1 \tan r_2 = \frac{n^3}{\sin s \sin(s-a) \sin(s-b)} = \frac{n^3 \sin(s-c)}{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}$$
$$= n \sin(s-c) \quad \ldots\ldots\ldots\ldots \quad (1)$$

Similarly, $\tan r \tan r_1 \tan r_3 = n \sin(s-b)$ .... (2)

$\tan r \tan r_2 \tan r_3 = n \sin(s-a)$ .... (3)

$\tan r_1 \tan r_2 \tan r_3 = n \sin s$ .... (4)

But, Davies's *Hutton*, vol. ii., p. 44.

$$\left.\begin{aligned} -\sin s + \sin(s-a) + \sin(s-b) + \sin(s-c) &= 2n \tan R \\ \sin s - \sin(s-a) + \sin(s-b) + \sin(s-c) &= 2n \tan R_1 \\ \sin s + \sin(s-a) - \sin(s-b) + \sin(s-c) &= 2n \tan R_2 \\ \sin s + \sin(s-a) + \sin(s-b) - \sin(s-c) &= 2n \tan R_3 \end{aligned}\right\} \ldots (A)$$

Hence, by (1, 2, 3, 4,) and (A,) we have

$$-\tan r\tan r_1\tan r_2+\tan r\tan r_1\tan r_3+\tan r\tan r_2\tan r_3+\ldots\ldots=2n^2\tan R_3\ldots(5)$$
$$\tan r\tan r_1\tan r_2-\tan r\tan r_1\tan r_3+\tan r\tan r_2\tan r_3+\ldots\ldots=2n^2\tan R_2\ldots(6)$$
$$\tan r\tan r_1\tan r_2+\tan r\tan r_1\tan r_3-\tan r\tan r_2\tan r_3+\ldots\ldots=2n^2\tan R_1\ldots(7)$$
$$\tan r\tan r_1\tan r_2+\tan r\tan r_1\tan r_3+\tan r\tan r_2\tan r_3-\ldots\ldots=2n^2\tan R\ldots(8)$$

By means of the values of cot R, cot R_1, &c., we also readily get the following corresponding relations:

$$-\cot R\cot R_1\cot R_2+\cot R\cot R_1\cot R_3+\cot R\cot R_2\cot R_3+\ldots=2N^2\cot r_3\ldots(9)$$
$$\cot R\cot R_1\cot R_2-\cot R\cot R_1\cot R_3+\cot R\cot R_2\cot R_3+\ldots=2N^2\cot r_2\ldots(10)$$
$$\cot R\cot R_1\cot R_2+\cot R\cot R_1\cot R_3-\cot R\cot R_2\cot R_3+\ldots=2N^2\cot r_1\ldots(11)$$
$$\cot R\cot R_1\cot R_2+\cot R\cot R_1\cot R_3+\cot R\cot R_2\cot R_3-\ldots=2N^2\cot r\ldots(12)$$

Adding equations (5, 6, 7, 8), and dividing by 2, we obtain the equality

$$\tan r\tan r_1\tan r_2+\tan r\tan r_1\tan r_3+\ldots=n^2(\tan R+\tan R_1+\tan R_2+\tan R_3)\ldots(13)$$

Taking (7) from (8), reducing and transposing, there results the relation

$$n^2\tan R_1+\tan r\tan r_2\tan r_3=n^2\tan R+\tan r_1\tan r_2\tan r_3\ldots\ldots(14)$$

Operating in a similar way with (5, 6) and (6, 8), we get

$$n^2\tan R_2+\tan r\tan r_1\tan r_3=n^2\tan R_3+\tan r\tan r_1\tan r_2\ldots(15)$$

and $n^2\tan R+\tan r_1\tan r_2\tan r_3=n^2\tan R_2+\tan r\tan r_1\tan r_3\ldots(16)$

∴ (14, 15, 16)

$$n^2\tan R+\tan r_1\tan r_2\tan r_3=n^2\tan R_1+\tan r\tan r_2\tan r_3=n^2\tan R_2+\tan r\tan r_1\tan r_3$$
$$=n^2\tan R_3+\tan r\tan r_1\tan r_2\ldots(17)$$

By repeated subtractions of equations (17), the six following relations are obtained:

$$n^2(\tan R-\tan R_1)=\tan r_2\tan r_3(\tan r-\tan r_1)\ldots\ldots(18)$$
$$n^2(\tan R-\tan R_2)=\tan r_1\tan r_3(\tan r-\tan r_2)\ldots\ldots(19)$$
$$n^2(\tan R-\tan R_3)=\tan r_1\tan r_2(\tan r-\tan r_3)\ldots\ldots(20)$$
$$n^2(\tan R_1-\tan R_2)=\tan r\tan r_3(\tan r_1-\tan r_2)\ldots\ldots(21)$$
$$n^2(\tan R_1-\tan R_3)=\tan r\tan r_2(\tan r_1-\tan r_3)\ldots\ldots(22)$$
$$n^2(\tan R_2-\tan R_3)=\tan r\tan r_1(\tan r_2-\tan r_3)\ldots\ldots(23)$$

Again, square each of the equations (5, 6, 7, 8) and add the results, then we get

$$\tan^2 r\tan^2 r_1\tan^2 r_2+\tan^2 r\tan^2 r_1\tan^2 r_3+\ldots=n^4(\tan^2 R+\tan^2 R_1+\tan^2 R_2+\tan^2 R_3)$$
$$\ldots\ldots(24)$$

Take the square of (7) from that of (8), and we have, after reduction,

$$n^4(\tan^2 R-\tan^2 R_1)=(\tan r-\tan r_1)(\tan r_2+\tan r_3)\tan r\tan r_1\tan r_2\tan r_3\ldots(25)$$

In a similar way (5, 6) give

$$n^4(\tan^2 R_2-\tan^2 R_3)=(\tan r+\tan r_1)(\tan r_2-\tan r_3)\tan r\tan r_1\tan r_2\tan r_3\ldots(26)$$

∴ (25, 26)

$$n^8(\tan^2 R-\tan^2 R_1)(\tan^2 R_2-\tan^2 R_3)$$
$$=(\tan^2 r-\tan^2 r_1)(\tan^2 r_2-\tan^2 r_3)\tan^2 r\tan^2 r_1\tan^2 r_2\tan^2 r_3\ldots\ldots(27)$$

Proceeding in like manner with (9, 10, 11, 12), we get, corresponding to the last fifteen, the following equalities:

$$\cot R\cot R_1\cot R_2+\cot R\cot R_1\cot R_3+\ldots=N^2(\cot r+\cot r_1+\cot r_2+\cot r_3)\quad\ldots(28)$$

$$N^2\cot r_1+\cot R\cot R_2\cot R_3=N^2\cot r+\cot R_1\cot R_2\cot R_3\quad\ldots(29)$$

$$N^2\cot r_2+\cot R\cot R_1\cot R_3=N^2\cot r_3+\cot R\cot R_1\cot R_2\quad\ldots(30)$$

$$N^2\cot r+\cot R_1\cot R_2\cot R_3=N^2\cot r_2+\cot R\cot R_1\cot R_3\quad\ldots(31)$$

$$N^2\cot r+\cot R_1\cot R_2\cot R_3=N^2\cot r_1+\cot R\cot R_2\cot R_3=N^2\cot r_2+\cot R\cot R_1\cot R_3$$
$$=N^2\cot r_3+\cot R\cot R_1\cot R_2\quad\ldots(32)$$

$$N^2(\cot r-\cot r_1)=\cot R_2\cot R_3(\cot R-\cot R_1)\quad\ldots(33)$$

$$N^2(\cot r-\cot r_2)=\cot R_1\cot R_3(\cot R-\cot R_2)\quad\ldots(34)$$

$$N^2(\cot r-\cot r_3)=\cot R_1\cot R_2(\cot R-\cot R_3)\quad\ldots(35)$$

$$N^2(\cot r_1-\cot r_2)=\cot R\cot R_3(\cot R_1-\cot R_2)\quad\ldots(36)$$

$$N^2(\cot r_1-\cot r_3)=\cot R\cot R_2(\cot R_1-\cot R_3)\quad\ldots(37)$$

$$N^2(\cot r_2-\cot r_3)=\cot R\cot R_1(\cot R_2-\cot R_3)\quad\ldots(38)$$

$$\cot^2R\cot^2R_1\cot^2R_2+\cot^2R\cot^2R_1\cot^2R_3+\ldots=N^4(\cot^2r+\cot^2r_1+\cot^2r_2+\cot^2r_3)$$
$$\ldots(39)$$

$$N^4(\cot^2r-\cot^2r_1)=(\cot R-\cot R_1)(\cot R_2+\cot R_3)\cot R\cot R_1\cot R_2\cot R_3\quad\ldots(40)$$

$$N^4(\cot^2r_2-\cot^2r_3)=(\cot R+\cot R_1)(\cot R_2-\cot R_3)\cot R\cot R_1\cot R_2\cot R_3\quad\ldots(41)$$

$$N^8(\cot^2r-\cot^2r_1)(\cot^2r_2-\cot^2r_3)$$
$$=(\cot^2R-\cot^2R_1)(\cot^2R_2-\cot^2R_3)\cot^2R\cot^2R_1\cot^2R_2\cot^2R_3\quad\ldots(42)$$

In the "Gentleman's Diary," for 1837, Mr. Rutherford gave a set of equations, which appear also to admit of many very elegant deductions. We shall add a few of them.

The equations to which we allude are those marked (B) which follow:

$$\left.\begin{aligned}-\cot r+\cot r_1+\cot r_2+\cot r_3&=2\tan R\\ \cot r-\cot r_1+\cot r_2+\cot r_3&=2\tan R_1\\ \cot r+\cot r_1-\cot r_2+\cot r_3&=2\tan R_2\\ \cot r+\cot r_1+\cot r_2-\cot r_3&=2\tan R_3\end{aligned}\right\}\quad\ldots(B)$$

Adding these equations, and dividing by 2, we get

$$\cot r+\cot r_1+\cot r_2+\cot r_3=\tan R+\tan R_1+\tan R_2+\tan R_3\quad\ldots(43)$$

Square each of the equations (B) and add the results, then we have

$$\cot^2r+\cot^2r_1+\cot^2r_2+\cot^2r_3=\tan^2R+\tan^2R_1+\tan^2R_2+\tan^2R_3\quad\ldots(44)$$

Of the same equations (B), take the square of the second from that of the first, and also the square of the first from that of the third, and we get

$$\tan^2R-\tan^2R_1=(\cot r-\cot r_1)(\cot r_2+\cot r_3)\quad\ldots(45)$$

$$\text{and, }\tan^2R_2-\tan^2R_3=(\cot r+\cot r_1)(\cot r_2-\cot r_3)\quad\ldots(46)$$

$\therefore$ (45, 46)

$$(\tan^2R-\tan^2R_1)(\tan^2R_2-\tan^2R_3)=(\cot^2r-\cot^2r_1)(\cot^2r_2-\cot^2r_3)\quad\ldots(47)$$

And since $\tan^2R=\sec^2R-1$, &c.; $\cot^2r=\operatorname{cosec}^2r-1$, &c.; hence (44) becomes

$$\sec^2R+\sec^2R_1+\sec^2R_2+\sec^2R_3=\operatorname{cosec}^2r+\operatorname{cosec}^2r_1+\operatorname{cosec}^2r_2+\operatorname{cosec}^2r_3\quad\ldots(48)$$

And so on in like manner we get other relations equally curious.

That marked (43) is already known.

ON THE

DETERMINATION OF THE MAGNITUDE AND POSITION OF A SPHERE TANGENT TO FOUR GIVEN SPHERES IN MUTUAL CONTACT.

First Investigation.

[*Mr. Rutherford.*]

The determination of the magnitude and position of a sphere tangent to four given spheres, has occupied the attention of several of the most illustrious continental geometers. Fermat was the first who effected the resolution of the problem, and, more recently, several very direct and elegant solutions, both geometrical and analytical, have been given in the *Journal de l'Ecole Polytechnique*, 17e cahier, p. 129, and in the *Correspondance sur cette école*, tome ii., p. 425. Other solutions are to be found in Hachette's *Descriptive Geometry*, pp. 160 and 265, and in Gergonne's *Annales de Mathematique*, p. 349. I am not, however, aware that any of the solutions which have been given of this interesting problem has any resemblance to that which is now presented to the perusal of the readers of the "Mathematician," and, so far as I know, neither the expression for the distance between the centres of the inscribed and circumscribed tangent spheres, nor the beautiful relation among the several radii which Mr. Woolhouse has given in the Lady's and Gentleman's Diary for the year 1843, has ever before been published.

Four spheres are placed in mutual contact, and another is described so as to be touched by all four externally: to find the magnitude and position of the inscribed tangent sphere.

Let c_1, c_2, c_3, c_4, c_5, be the centres, r_1, r_2, r_3, r_4, r_5, the corresponding radii of the five spheres, c_5 and r_5 being the centre and radius of the required sphere. Take c_1 for the origin of co-ordinates, the line joining $c_1\, c_2$ for the axis of x, and the plane through $c_1\, c_2\, c_3$ for the plane of xy. Denote the co-ordinates of the centres c_2, c_3, c_4, c_5, by $\alpha\, 0\, 0$, $\alpha_1\, \beta_1\, 0$, $\alpha_2\, \beta_2\, \gamma_2$, $x\, y\, z$ respectively, and the contact of the five spheres will afford the subsequent ten equations, viz.

$$\alpha^2=(r_1+r_2)^2\ldots.(1) \qquad \alpha_2^2+\beta_2^2+\gamma_2^2=(r_1+r_4)^2\ldots(4)$$
$$\alpha_1^2+\beta_1^2=(r_1+r_3)^2\ldots.(2) \qquad (\alpha-\alpha_2)^2+\beta_2^2+\gamma_2^2=(r_2+r_4)^2\ldots(5)$$
$$(\alpha-\alpha_1)^2+\beta_1^2=(r_2+r_3)^2\ldots.(3) \qquad (\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2+\gamma_2^2=(r_3+r_4)^2\ldots(6)$$
$$x^2+y^2+z^2=(r_1+r_5)^2\ldots\ldots.(7)$$
$$(\alpha-x)^2+y^2+z^2=(r_2+r_5)^2\ldots\ldots.(8)$$
$$(\alpha_1-x)^2+(\beta_1-y)^2+z^2=(r_3+r_5)^2\ldots\ldots.(9)$$
$$(\alpha_2-x)^2+(\beta_2-y)^2+(\gamma_2-z)^2=(r_4+r_5)^2\ldots\ldots.(10)$$

The last four of these equations (7, 8, 9, 10) are sufficient for the deter-

mination of the values of the four unknown quantities x, y, z, r_5, in terms of $\alpha, \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_2$. It is very obvious that, since the equations (4, 5, 6) are precisely of the same form as the equations (7, 8, 9), the values of x, y, z, deduced from the latter, must have the same form as the values of $\alpha_2, \beta_2, \gamma_2$ from the former, with this difference only, that the radius of r_5 in the values of x, y, z, will occupy the place of r_4 in the values of $\alpha_2, \beta_2, \gamma_2$, and consequently the values of the co-ordinates $\alpha_2, \beta_2, \gamma_2$, will at once furnish the values of the co-ordinates x, y, z, by the simple change of the radius. Hence, if we subtract

(3) from the sum of (1) and (2); (5) from the sum of (1) and (4);

(6) from the sum of (2) and (4); (10) from the sum of (4) and (7);

and divide each remainder by 2, we shall have, in consequence of the preceding observation, the following symmetrical equations:

$$\alpha\alpha_1 = r^2_1 + r_1r_2 + r_1r_3 - r_2r_3 \ldots\ldots (11)$$
$$\alpha\alpha_2 = r^2_1 + r_1r_2 + r_1r_4 - r_2r_4 \ldots\ldots (12)$$
$$\alpha x = r^2_1 + r_1r_2 + r_1r_5 - r_2r_5 \ldots\ldots (13)$$
$$\alpha_1\alpha_2 + \beta_1\beta_2 = r^2_1 + r_1r_3 + r_1r_4 - r_3r_4 \ldots\ldots (14)$$
$$\alpha_1 x + \beta_1 y = r^2_1 + r_1r_3 + r_1r_5 - r_3r_5 \ldots\ldots (15)$$
$$\alpha_2 x + \beta_2 y + \gamma_2 z = r^2_1 + r_1r_4 + r_1r_5 - r_4r_5 \ldots\ldots (16)$$

and from the first three of these equations we have

$$\alpha^2\alpha_1\alpha_2 = (r^2_1 + r_1r_2 + r_1r_3 - r_2r_3)(r^2_1 + r_1r_3 + r_1r_4 - r_2r_4) \ldots (17)$$
$$\alpha^2\alpha_1 x = (r^2_1 + r_1r_2 + r_1r_3 - r_2r_3)(r^2_1 + r_1r_2 + r_1r_5 - r_2r_5) \ldots (18)$$
$$\alpha^2\alpha_2 x = (r^2_1 + r_1r_2 + r_1r_4 - r_2r_4)(r^2_1 + r_1r_2 + r_1r_5 - r_2r_5) \ldots (19)$$

Again, because the line $c_1\,c_2$ is the common intersection of the planes through $c_1c_2c_3$, $c_1c_2c_4$, $c_1c_2c_5$, it is obvious that β^2_1, $\beta^2_2 + \gamma^2_2$, and $y^2 + z^2$, which are the squares of the distances of the centres c_3, c_4, c_5, from the line $c_1\,c_2$, must be of the same form with respect to the corresponding radii r_3, r_4, r_5; consequently, by means of the equations (2) and (11), (4) and (12), (7) and (13), we readily obtain the equations below:

$$\alpha^2\beta^2_1 = \alpha^2(r_1 + r_3)^2 - \alpha^2\alpha^2_1 = 4r_1r_2r_3(r_1 + r_2 + r_3) \ldots (20)$$
$$\alpha^2(\beta^2_2 + \gamma^2_2) = \alpha^2(r_1 + r_4)^2 - \alpha^2\alpha^2_2 = 4r_1r_2r_4(r_1 + r_2 + r_4) \ldots (21)$$
$$\alpha^2(y^2 + z^2) = \alpha^2(r_1 + r_5)^2 - a^2x^2 = 4r_1r_2r_5(r_1 + r_2 + r_5) \ldots (22)$$

Multiplying each of the equations (14, 15, 16), by α^2, and subtracting the corresponding equations (17, 18, 19), we get the resulting equations

$$\alpha^2\beta_1\beta_2 = 2r^2_1(r_2r_3 + r_2r_4 - r_3r_4) + 2r^2_2(r_1r_3 + r_1r_4 - r_3r_4) \ldots (23)$$
$$\alpha^2(\beta_2 y + \gamma_2 z) = 2r^2_1(r_2r_4 + r_2r_5 - r_4r_5) + 2r^2_2(r_1r_4 + r_1r_5 - r_4r_5) \ldots (24)$$

Divide the square of (23) by (20), and extract the root of the quotient; then, writing y and r_5 for β_2 and r_4, we have at once the two equations

$$\alpha\beta_2\sqrt{r_1r_2r_3(r_1+r_2+r_3)} = r^2_1(r_2r_3+r_2r_4-r_3r_4) + r^2_2(r_1r_3 + r_1r_4 - r_3r_4) \ldots (25)$$
$$\alpha y\sqrt{r_1r_2r_3(r_1+r_2+r_3)} = r^2_1(r_2r_3+r_2r_5-r_3r_5) + r^2_2(r_1r_3 + r_1r_5 - r_3r_5) \ldots (26)$$

which determine the values of the co-ordinates β_2 and y of the centres c_4 and c_5. From these equations we have the values of $\alpha^2\beta^2_2$ and α^2y^2, which being subtracted respectively from (21) and (22), we obtain, after reducing and

dividing by α^2, or its equivalent $(r_1 + r_2)^2$, the following equations for the determination of the co-ordinates γ_2 and z, viz.

$$\gamma^2_2 = \frac{2r_1r_2r_3r_4(r_1 + r_2)(r_3 + r_4) - r^2_1r^2_2(r_3 - r_4)^2 - r^2_3r^2_4(r_1 - r_2)^2}{r_1r_2r_3(r_1 + r_2 + r_3)} \quad ..(27)$$

$$z^2 = \frac{2r_1r_2r_3r_5(r_1 + r_2)(r_3 + r_5) - r^2_1r^2_2(r_3 - r_5)^2 - r^2_3r^2_5(r_1 - r_2)^2}{r_1r_2r_3(r_1 + r_2 + r_3)} \quad ..(28)$$

These last equations will assume a different form by dividing the numerator and denominator of (27) by $r^2_1r^2_2r^2_3r^2_4$, and those of (28) by $r^2_1r^2_2r^2_3r^2_5$; hence we have

$$\frac{1}{r^2_4}\left(\frac{1}{r_1r_2} + \frac{1}{r_1r_3} + \frac{1}{r_2r_3}\right) \cdot \gamma^2_2 = \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)^2$$
$$- 2\left(\frac{1}{r^2_1} + \frac{1}{r^2_2} + \frac{1}{r^2_3} + \frac{1}{r^2_4}\right) \ldots\ldots (27a)$$

$$\frac{1}{r^2_5}\left(\frac{1}{r_1r_2} + \frac{1}{r_1r_3} + \frac{1}{r_2r_3}\right) \cdot z^2 = \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_5}\right)^2$$
$$- 2\left(\frac{1}{r^2_1} + \frac{1}{r^2_2} + \frac{1}{r^2_3} + \frac{1}{r^2_5}\right) \ldots\ldots (28a)$$

But these equations may be still further simplified in the following manner: Let us put

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{s} \;..(29) \text{ and } \frac{1}{r_1r_2} + \frac{1}{r_1r_3} + \frac{1}{r_2r_3} = \frac{1}{p^2} \ldots\ldots (30)$$

$$\therefore \frac{1}{r^2_1} + \frac{1}{r^2_2} + \frac{1}{r^2_3} = \frac{1}{s^2} - \frac{2}{p^2} \ldots\ldots (31)$$

then (27*a*) and (28*a*) will be transformed into the following equations:

$$s^2\gamma^2_2 = 4s^2r^2_4 - p^2(s - r_4)^2 = \{(2s + p)r_4 - sp\} \cdot \{(2s - p)r_4 + sp\} \;..(27b)$$

$$s^2z^2 = 4s^2r^2_5 - p^2(s - r_5)^2 = \{(2s + p)r_5 - sp\} \cdot \{(2s - p)r_5 + sp\} \;..(28b)$$

Again, from (24) subtract the product of (25) by (26), and simplify the remainder by reducing and dividing by α^2 or $(r_1 + r_1)^2$, and we have ultimately

$$\gamma_2 z = \frac{r_1r_2r_3(r_1r_2 + r_1r_3 + r_2r_3)(r_4 + r_5) - (r^2_1r^2_2 + r^2_1r^2_3 + r^2_2r^2_3)r_4r_5 - r^2_1r^2_2r^2_3}{r_1r_2r_3(r_1 + r_2 + r_3)} \;..(32)$$

which, in the more convenient notation of (29) and (30), becomes

$$s^2\gamma_2 z = 2s^2r_4r_5 - p^2(s - r_4)(s - r_5) \ldots\ldots\ldots (33)$$

Lastly, if we equate the product of (27*b*) by (28*b*) to the square of (33), we get

$$\frac{1}{r^2_5} - \left(\frac{1}{s} + \frac{1}{r_4}\right) \cdot \frac{1}{r_5} = \frac{3}{p^2} - \frac{1}{s^2} + \frac{1}{r_4} \cdot \frac{1}{s} - \frac{1}{r^2_4} \;\ldots (34)$$

which, by restoring the values of the reciprocals of s and p^2, gives, finally,

$$\frac{1}{r^2_5} - \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) \cdot \frac{1}{r_5}$$
$$= \frac{1}{r_1r_2} + \frac{1}{r_1r_3} + \frac{1}{r_2r_3} + \frac{1}{r_1r_4} + \frac{1}{r_2r_4} + \frac{1}{r_3r_4} - \frac{1}{r^2_1} - \frac{1}{r^2_2} - \frac{1}{r^2_3} - \frac{1}{r^2_4} \;..(35)$$

and from this equation the magnitude of the sphere touched externally by the four given spheres, *can readily be found.* In order to abridge, let us assume

$$\left.\begin{array}{r}\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}+\dfrac{1}{r_4}=\dfrac{2}{m}\cdots\cdots\cdots\cdots\\[2ex] \dfrac{1}{r_1r_2}+\dfrac{1}{r_1r_3}+\dfrac{1}{r_2r_3}+\dfrac{1}{r_1r_4}+\dfrac{1}{r_2r_4}+\dfrac{1}{r_3r_4}=\dfrac{1}{n^2}\cdots\cdots\cdots\cdots\\[2ex] \dfrac{1}{r^2_1}+\dfrac{1}{r^2_2}+\dfrac{1}{r^2_3}+\dfrac{1}{r^2_4}=\dfrac{1}{q^2}\cdots\cdots\cdots\cdots\end{array}\right\}(36)$$

and substitute these new quantities in (35); then resolving that equation, we get

$$\frac{1}{r_5}=\frac{1}{m}\pm\left\{\frac{1}{m^2}+\frac{1}{n^2}-\frac{1}{q^2}\right\}^{\frac{1}{2}}\cdots\cdots(37)$$

Thus the magnitude and position of the sphere touching four spheres placed in mutual contact, are completely determined, the radius being found in (37), and thence the values of x, y, z, the co-ordinates of its centre from the equations (13), (26), and (28).

The magnitude and position of the sphere touched *internally* by four spheres in mutual contact, may be deduced at once from the preceding investigation.

For in this case we have simply to write in the equations (7, 8, 9, 10) the *difference* of the radii instead of their *sum;* consequently, if c_6 and r_6 denote the centre and radius of the sixth sphere touched internally by the four given ones, we have only to write $-r_6$ for r_5 in equation (35), and resolving as before, we get

$$\frac{1}{r_6}=-\frac{1}{m}+\left\{\frac{1}{m^2}+\frac{1}{n^2}-\frac{1}{q^2}\right\}^{\frac{1}{2}}\cdots\cdots\cdots\cdots(38)$$

which determines the magnitude of the circumscribed tangent sphere, and if $x_1\ y_1\ z_1$ denote the co-ordinates of its centre c_6, their values will be found from the same equations (13), (26), (28), from which $x\,y\,z$ were found, by writing in these x_1, y_1, z_1, and $-r_6$ for x, y, z, and r_5.

Subtracting (38) from (37) we get the following very remarkable and elegant theorem, the reciprocal of m being replaced by its value in (36), viz.

$$\frac{1}{r_5}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}+\frac{1}{r_6}\cdots\cdots\cdots(39)$$

which may be thus enunciated:

If four spheres be placed in mutual contact, and two others be described to be touched by all four internally and externally; then the reciprocal of the radius of the least or inscribed sphere will be equal to the sum of the reciprocals of the other five.

Let us now take the product of the equations (37) and (38); then we have

$$\frac{1}{r_5r_6}=\frac{1}{n^2}-\frac{1}{q^2},\text{ or}$$

$$\frac{1}{r_5r_6}=\frac{1}{r_1r_2}+\frac{1}{r_1r_3}+\frac{1}{r_2r_3}+\frac{1}{r_1r_4}+\frac{1}{r_2r_4}+\frac{1}{r_3r_4}-\frac{1}{r^2_1}-\frac{1}{r^2_2}-\frac{1}{r^2_3}-\frac{1}{r^2_4}\ (40)$$

$$\text{but since}\quad\frac{1}{r_5}-\frac{1}{r_6}=\frac{2}{m}$$

$$\therefore\ \frac{1}{r_5}+\frac{1}{r_6}=2\left\{\frac{1}{m^2}+\frac{1}{n^2}-\frac{1}{q^2}\right\}^{\frac{1}{2}}\cdots\cdots\cdots(41)$$

If the radii of the four given spheres be all equal, the preceding formulæ for the radii of the inscribed and circumscribed tangent spheres will become

$$\frac{1}{r^2_5} - \frac{4}{r_1 r_5} = \frac{2}{r^2_1} \text{ or } \frac{1}{r_5} = \frac{2 \pm \sqrt{6}}{r_1} \ldots\ldots (42)$$

$$\frac{1}{r^2_6} + \frac{4}{r_1 r_6} = \frac{2}{r^2_1} \text{ or } \frac{1}{r_6} = \frac{-2 \pm \sqrt{6}}{r_1} \ldots . (43)$$

and the co-ordinates of their centres, in this case coincident, are as below:

$$x = r_1;\ y = \frac{r_1}{3}\sqrt{3};\ z = \frac{r_1}{6}\sqrt{6} \ldots\ldots\ldots (44)$$

Taking the product of (42) and (43) we get a remarkable relation, viz.:

$$\frac{1}{r_5 r_6} = \frac{2}{r^2_1},\ \text{or } 2r_5 r_6 = r^2_1 \ldots\ldots\ldots (45)$$

that is, the square of the radius of either of the equal spheres is equal to twice the product of the radii of the inscribed and circumscribed tangent spheres.

And if the magnitudes of the three spheres c_1, c_2, c_3, are all equal, then we have for the determination of the reciprocals of the two radii, the equations

$$\frac{1}{r^2_5} - \left(\frac{3}{r_1} + \frac{1}{r_4}\right) \cdot \frac{1}{r_5} = \left(\frac{3}{r_1} - \frac{1}{r_4}\right) \cdot \frac{1}{r_4} \ldots\ldots\ldots (46)$$

$$\frac{1}{r^2_6} + \left(\frac{3}{r_1} + \frac{1}{r_4}\right) \cdot \frac{1}{r_6} = \left(\frac{3}{r_1} - \frac{1}{r_4}\right) \cdot \frac{1}{r_4} \ldots\ldots\ldots (47)$$

and the co-ordinates of their centres are found to be as in the equations below:

$$x = r_1,\ y = \frac{r_1\sqrt{3}}{3},\ z^2 = (r_1 + r_5)^2 - \frac{4}{3} r^2_1 \ldots\ldots\ldots (48)$$

$$x_1 = r_1,\ y_1 = \frac{r_1\sqrt{3}}{3},\ z^2_1 = (r_1 - r_6)^2 - \frac{4}{3} r^2_1 \ldots\ldots\ldots (49)$$

where the values of z and z_1 will be known by substituting for r_5 and r_6 their values deduced from the preceding equations (46) and 47).

To determine the distance between the centres of the inscribed and circumscribed tangent spheres, we have, by (13),

$$ax = r^2_1 + r_1 r_2 + r_1 r_5 - r_2 r_5,$$

$$ax_1 = r^2_1 + r_1 r_2 - r_1 r_6 + r_2 r_6;$$

$$\therefore a(x - x_1) = (r_1 - r_2)(r_5 + r_6) \ldots\ldots\ldots (50)$$

Again, by (26), we have

$$ay\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)} = r^2_1 (r_2 r_3 + r_2 r_5 - r_3 r_5) + r^2_2 (r_1 r_3 + r_1 r_5 - r_3 r_5),$$

$$ay_1\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)} = r^2_1 (r_2 r_3 - r_2 r_6 + r_3 r_6) + r^2_2 (r_1 r_3 - r_1 r_6 + r_3 r_6);$$

$$\therefore a(y - y_1) = \frac{\{r_1 r_2 (r_1 + r_2) - r_3 (r^2_1 + r^2_2)\}(r_5 + r_6)}{\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}} \ldots (51)$$

Whence, by adding the squares of the equations (50) and (51), reducing and dividing both sides by a^4, we obtain

$$\frac{(x-x_1)^2+(y-y_1)^2}{(r_5+r_6)^2}=\frac{r^2_1r^2_2+r^2_1r^2_3+r^2_2r^2_3}{r_1r_2r_3(r_1+r^2+r_3)}-1=\frac{p^2}{s^2}-3\,..(52)$$

from which the distance of the centres projected on the plane of xy can be found.

Moreover, by (33), we have for the values of z and z_1 the two equations

$$s^2\gamma_2 z=2\,s^2r_4r_5-p^2\{s^2-(r_4+r_5)\,s+r_4r_5\},$$

$$s^2\gamma_2 z_1=-\,2s^2r_4r_6-p^2\{s^2-(r_4-r_6)s-r_4r_6\};$$

$$\therefore\ s^2\gamma_2(z-z_1)=\{2s^2r_4+p^2(s-r_4)\}(r_5+r_6);$$

$$\therefore\ \frac{(z-z_1)^2}{(r_5+r_6)^2}=\frac{\{2s^2r_4+p^2\,(s-r_4)\}^2}{s^4\gamma^2_2}=\frac{\{2s^2r_4+p^2(s-r_4)\}^2}{s^2\{4s^2r^2_4-p^2(s-r_4)^2\}}\ \ldots.(53)$$

Now if d denote the distance in space between the centres of the two spheres, we have

$$d^2=(x-x_1)^2+(y-y_1)^2+(z-z_1)^2,$$

and, therefore, by substituting in this the sum of equations (52) and (53), we get

$$\frac{d^2}{(r_5+r_6)^2}=\frac{3p^2(s+r_4)^2-8sr_4(p^2+sr_4)}{4s^2r^2_4-p^2(s-r_4)^2};$$

which, by expanding the right hand member, and dividing numerator and denominator by $p^2s^2r^2_4$, becomes in the notation of (36)

$$\frac{d^2}{(r_5+r_6)^2}=\frac{\dfrac{3}{q^2}-\dfrac{3}{m^2}}{\dfrac{1}{m^2}+\dfrac{1}{n^2}-\dfrac{1}{q^2}}.$$

But the value of $(r_5+r_6)^2$, which is easily obtained from (40) and (41), is

$$(r_5+r_6)^2=\frac{\dfrac{4}{m^2}+\dfrac{4}{n^2}-\dfrac{4}{q^2}}{\left(\dfrac{1}{n^3}-\dfrac{1}{q^2}\right)^2}\quad\therefore\quad d^2=\frac{12\left(\dfrac{1}{q^2}-\dfrac{1}{m^2}\right)}{\left(\dfrac{1}{n^2}-\dfrac{1}{q^2}\right)^2},$$

$$\text{or } d=\frac{2\sqrt{3}\left\{\dfrac{1}{r^2_1}+\dfrac{1}{r^2_2}+\dfrac{1}{r^2_3}+\dfrac{1}{r^2_4}-\dfrac{1}{4}\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}+\dfrac{1}{r_4}\right)^2\right\}^{\frac{1}{2}}}{\dfrac{1}{r_1r_2}+\dfrac{1}{r_1r_3}+\dfrac{1}{r_2r_3}+\dfrac{1}{r_1r_4}+\dfrac{1}{r_2r_4}+\dfrac{1}{r_3r_4}-\left(\dfrac{1}{r^2_1}+\dfrac{1}{r^2_2}+\dfrac{1}{r^2_3}+\dfrac{1}{r^2_4}\right)}$$

$$\ldots\ldots..(54)$$

which determines the distance between the centres of the inscribed and circumscribed tangent spheres, as a function of the radii of the given spheres. When the four radii, r_1, r_2, r_3, r_4, are all equal; then $d=0$, and when r_1, r_2, r_3 are equal; then we get

$$d=3\left(\frac{1}{r_1}-\frac{1}{r_4}\right)\div\frac{1}{r_4}\left(\frac{3}{r_1}-\frac{1}{r_4}\right)\ldots.(57)$$

We have thus resolved this interesting problem in all its generality, and from what has been done we may readily resolve the following problem, viz.:

Three spheres are placed in mutual contact, to find the magnitude and position of another sphere which shall touch the three given spheres and the tangent plane.

Since a plane may be conceived to be a sphere, whose radius is indefinitely great, we have simply to make r_4 infinite in the preceding formulæ and we shall obtain the radius and the co-ordinates of the centre of the required sphere. Thus, in eq. (35), if r_4 is infinite, we have

$$\frac{1}{r_5^2}-\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)\frac{1}{r_5}=\frac{1}{r_1r_2}+\frac{1}{r_1r_3}+\frac{1}{r_2r_3}-\left(\frac{1}{r^2_1}+\frac{1}{r^2_2}+\frac{1}{r^2_3}\right)\quad(58)$$

or, in the notation of (36), we get

$$\frac{1}{r_5}=\frac{1}{m_1}\pm\left\{\frac{1}{m^2_1}+\frac{1}{n^2_1}-\frac{1}{q^2_1}\right\}^{\frac{1}{2}}\quad\ldots\ldots(59)$$

where m_1, n_1, q_1 are what m, n, q become when in their values r_4 is taken indefinitely great. The co-ordinates of the centre of the sphere are obtained from the same formulæ as were employed for finding the co-ordinates of the centres in the general investigation.*

Second Investigation.

[*Mr. Fenwick.*]

Though this problem has engaged the attention of Carnot, Feuerbach and some other eminent continental mathematicians, the radii of the fifth and sixth spheres have not yet been given, I believe, in terms of the four given radii. This, however, which indeed presents the chief difficulty in the solution, is effected in the following investigation; and thus, whilst the problem, with respect to finding the radii of the fifth and sixth spheres, is completely resolved, we are able, at the same time, to prove the remarkable relation amongst the reciprocals of the six radii, due to Mr. Woolhouse. The sixth sphere is that which is touched internally by the four given spheres. Let three rectangular planes pass through the centre of one of the four spheres in contact, so that the co-ordinates of their centres may be

$$0\,0\,0;\quad \alpha\,0\,0;\quad \alpha_1\,\beta_1\,0;\quad \alpha_2\,\beta_2\,\gamma_2;$$

and radii r, r_1, r_2, r_3. Let also $x\,y\,z$ be the co-ordinates of the centre of that sphere (the fifth) which is touched by these externally, and R its radius.

* The double sign in equation (37) will give the reciprocal of the radii of both the inscribed and circumscribed tangent spheres, by considering the radius r_5 to be positive in the former case and negative in the latter. Thus, if r_5 be positive and the + sign be taken, we shall have the reciprocal of the radius of the sphere touched externally by all four, and if r_5 be negative, the — sign will give eq. (38) for the reciprocal of the radius of the sphere touched internally by the four given spheres.

Then ρ being the distance between the centres of the first and fifth spher
we have the following equations:—

$$\rho^2 = x^2 + y^2 + z^2 \dots\dots\dots\dots\dots\dots(1)$$
$$(\rho - \delta)^2 = (x - \alpha)^2 + y^2 + z^2 \dots\dots\dots\dots\dots(2)$$
$$(\rho - \delta_1)^2 = (x - \alpha_1)^2 + (y - \beta_1)^2 + z^2 \dots\dots\dots(3)$$
$$(\rho - \delta_2)^2 = (x - \alpha_2)^2 + (y - \beta_2)^2 + (z - \gamma_2)^2 \dots(4)$$

where, $\delta = r_1 - r_2$; $\delta_1 = r_1 - r_3$; $\delta_2 = r_1 - r_4$.

These equations are sufficient to determine the radius of the fifth sphe
but the following are employed with advantage in reduction: viz.

$(r_1+r_2)^2=\alpha^2 \dots\dots\dots(5)$ $(r_2+r_3)^2=(\alpha-\alpha_1)^2+\beta^2_1 \dots\dots(8$
$(r_1+r_3)^2=\alpha^2_1+\beta^2_1 \dots\dots(6)$ $(r_2+r_4)^2=(\alpha-\alpha_2)^2+\beta^2_2+\gamma^2_2 \dots(9$
$(r_1+r_4)^2=\alpha^2_2+\beta^2_2+\gamma^2_2 \dots(7)$ $(r_3+r_4)^2=(\alpha_1-\alpha_2)^2+(\beta_1-\beta_2)^2+\gamma^2_2(1$

Taking (1) from (2) and reducing by (5), we readily get

$$\rho\delta = \alpha x - 2r_1 r_2 \dots\dots(11)$$

Similarly, equations (1, 3, 6), and (1, 4, 7), give

$$\rho\delta_1 = \alpha_1 x + \beta_1 y - 2r_1 r_3 \dots\dots(12)$$
$$\text{and } \rho\delta_2 = \alpha_2 x + \beta_2 y + \gamma_2 z - 2r_1 r_4 \dots(13)$$

If, now, we eliminate x from (12) by means of (11) and x,y from(13)
(11, 12); and put

$m=\alpha$, $m_1=\alpha\beta_1$, $m_2=\alpha\beta_1\gamma_2$,
$n=\delta$, $n_1=\delta_1\alpha-\delta a_1$, $n_2=a\beta_1\delta_2-(a_2\beta_1-a_1\beta_2)\delta-a\beta_2\delta_1$,
$p=2r_1r_2$, $p_1=2r_1(r_3a-r_2a_1)$, $p_2=2r_1\{a\beta_1r_4-(a_2\beta_1-a_1\beta_2)r_2-a\beta_2r_3\}$;

we shall have from (11, 12, 13)

$$x = \frac{p+n\rho}{m};\ y = \frac{p_1+n_1\rho}{m_1};\ z = \frac{p_2+n_2\rho}{m_2}.$$

Substituting these values of x, y, z, in (1); writing $(R + r_1)$ for ρ; a
solving for $\frac{1}{R}$; we get

$$\frac{1}{R} = -\frac{e}{f} \pm \frac{1}{f}\left\{e^2 + fh\right\}^{\frac{1}{2}} \dots(14),$$

where,

$$e = m^2_1 m^2_2 n\lambda + m^2 m^2_2 n_1 \lambda_1 + m^2 m^2_1 n_2 \lambda_2 - m^2 m^2_1 m^2_2 r_1 \dots\dots(15)$$
$$f = m^2_1 m^2_2 \lambda^2 + m^2 m^2_2 \lambda^2_1 + m^2 m^2_1 \lambda^2_2 - m^2 m^2_1 m^2_2 r^2_1 \dots\dots(16)$$
$$h = m^2 m^2_1 m^2_2 - (m^2_1 m^2_2 n^2 + m^2 m^2_2 n^2_1 + m^2 m^2_1 n^2_2) \dots\dots(17)$$

λ, λ_1, λ_2 being written for $(p + n\,r_1)$, $(p_1 + n_1\,r_1)$, and $(p_2 + n_2\,r$
respectively.

In a similar way we find the reciprocal of the sixth radius. It is

r_1, $$\frac{1}{R_1} = \frac{e}{f} \pm \frac{1}{f}\left\{e^2 + fh\right\}^{\frac{1}{2}} \dots(18)$$

ain, to express the co-ordinates in terms of the respective radii, put

$+ r_2 = a = \alpha$ $r_4 = a_1$,
$r_3 = b$, $r_4 = b_1$, $\frac{1}{2}(a + b + c) = r_1 + r_2 + r_3 = s$,
$r_3 = c$, $= c_1$, $\frac{1}{2}(a + b_1 + c_1) = r_1 + r_2 + r_4 = s_1$.

Then, from the triangles formed by joining the centres of the spheres whose radii are r_1, r_2, r_3, and r_1, r_2, r_4, respectively, we have

$$a_1 = \frac{a^2+b^2-c^2}{2a} = \frac{ar_1 + cr_3}{a} \quad \ldots.(19) \qquad a_2 = \frac{ar_1 + cr_4}{a} \quad \ldots\ldots(20)$$

$$\beta^2_1 = \frac{4s(s-a)(s-b)(s-c)}{a^2} = \frac{4sr_1r_2r_3}{a^2} \quad ..(21) \qquad \beta^2_2 + \gamma^2_2 = \frac{4s_1r_1r_2r_4}{a^2} \quad (22)$$

To express β_2 take (10) from (9), and reduce by (5, 6); then

$$\beta_1\beta_2 = (r_1 - r_4)(r_3 - r_2) + a_2(a - a_1) = \frac{2a\,a_1r_1r_2 - 2r_3r_4(r^2_1 + r^2_2)}{a^2} \quad ..(23)$$

$$\therefore\ \beta^2_2 = \frac{\beta^2_1\,\beta^2_2}{\beta^2_1} = \frac{\{a\,a_1r_1r_2 - r_3r_4(r^2_1+r^2_2)\}^2}{a^2s\,r_1r_2r_3} \quad \ldots\ldots\ldots.(24)$$

$$\text{and } \frac{\beta_2}{\beta_1} = \frac{\beta_1\beta_2}{\beta^2_1} = \frac{a\,a_1r_1r_2 - r_3r_4(r^2_1 + r^2_2)}{2s\,r_1r_2r_3} \quad \ldots\ldots\ldots.(25)$$

Hence (22, 24)

$$\gamma^2_2 = \frac{4s_1r_1r_2r_4}{a^2} - \beta^2_2 = \frac{2aa_1r_1r_2r_3r_4 - r^2_1r^2_2(r_4-r_3)^2 - r^2_3r^2_4(r_1-r_2)^2}{s\,r_1r_2r_3} \quad ..(26)$$

The following expressions are easily deduced:

$$m^2 = (r_1 + r_2)^2, \qquad m^2_1 = 4r_1r_2r_3(r_1 + r_2 + r_3),$$

$$n = r_1 - r_2, \qquad n_1 = \frac{2}{r_1 + r_2}\{r_1r_2(r_1 + r_2) - r_3(r^2_1 + r^2_2)\},$$

$$\lambda = r_1(r_1+r_2), \qquad \lambda_1 = 2\,r_1r_2r_3,$$

$$m^2_2 = 4\{2r_1r_2r_3r_4(r_1 + r_2)(r_3 + r_4) - r^2_1r^2_2(r_4 - r_3)^2 - r^2_3r^2_4(r_1 - r_2)^2\},$$

$$n_2 = \frac{\beta_1(r_1+r_2)}{r_1r_2r_3s}\{r_1r_2r_3^2(r_1 + r_2) - r^2_1r^2_2(r_4 - r_3) - r_4r^2_3(r^2_1 + r^2_2)\},$$

$$\lambda_2 = \frac{\beta_1(r_1+r_2)}{s}\{r_1r_2(r_4 - r_3) + r_3r_4(r_1 + r_2)\}.$$

The radii, then, of the fifth and sixth spheres, in terms of the four given radii, are completely determined.

Again, since

$$m^2_1m^2_2n\lambda - m^2m^2_1m^2_2r_1 = m^2_1m^2_2\{r_1(r^2_1 - r^2_2) - r_1(r_1+r_2)^2\} = -2r_1r_2a\,m^2_1m^2_2,$$

$$\text{and } m^2_1m^2_2\lambda^2 - m^2m^2_1m^2_2r^2_1 = m^2_1m^2_2\{r^2_1(r_1+r_2)^2 - r^2_1(r_1+r_2)^2\} = 0;$$

$$\therefore \frac{e}{f} = \frac{m^2m^2_2n_1\lambda_1 + m^2m^2_1n_2\lambda_2 - 2r_1r_2\,a\,m^2_1m^2_2}{m^2(m^2_2\lambda^2_1 + m^2_1\lambda^2_2)}$$

$$= \frac{m\,m^2_2n_1\lambda_1 + m\,m^2_1n_2\lambda_2 - 2\,r_1r_2\,m^2_1m^2_2}{m(m^2_2\lambda^2_1 + m^2_1\lambda^2_2)}$$

$$= -\frac{32r^2_1r^2_2r^2_3r_4\,a\,s(r_2r_3r_4 + r_1r_3r_4 + r_1r_2r_4 + r_1r_2r_3)}{64\,r^3_1r^3_2r^3_3r^2_4\,a\,s}$$

$$= -\frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right)\ldots.(27)$$

$\therefore$ (14, 18, 27)

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + \frac{1}{R_1} \quad \ldots.(28)$$

Or the reciprocal of the fifth radius is equal to the sum of the reciprocals of the other five.

This remarkable and elegant property has been proposed for demonstration in the "Diary" for the present year, by the Editor of that periodical.

We can find the radii of the fifth and sixth spheres by another method, thus:

If in (26) we write R for r_4, it will be obvious that the resulting equation will be the value of z^2;

$$\therefore \quad z^2 = \frac{2r_1r_2r_3\,\mathrm{R}(r_1+r_2)(\mathrm{R}+r_3) - r^2_1r^2_2(\mathrm{R}-r_3)^2 - \mathrm{R}^2r^2_3(r_1-r_2)^2}{s\,r_1r_2r_3} \quad ..(29)$$

$$\text{But, } z = \frac{p_2 + n_2\rho}{m_2} = \frac{p_2 + n_2(\mathrm{R}+r_1)}{m_2} \quad ....(30)$$

Hence, equating the square of (30) with (29), and solving for $\frac{1}{\mathrm{R}}$, we have

$$\frac{1}{\mathrm{R}} = -\frac{e_1}{f_1} \pm \frac{1}{f_1}\sqrt{e^2_1 + f_1 h_1} \quad ....(31),$$

where

$$e_1 = \frac{n_2\lambda_2}{m^2_2} - \frac{r_1r_2 + r_1r_3 + r_2r_3}{s} \quad ....(32)$$

$$f = \frac{\lambda^2_2}{m^2_2} + \frac{r_1r_2r_3}{s} \quad ....(33)$$

$$h_1 = \frac{2\,r_1r_2r_3\,s - (r^2_1r^2_2 + r^2_1r^2_3 + r^2_2r^2_3)}{r_1r_2r_3\,s} - \frac{n^2_2}{m^2_2} \quad ......(34)$$

$$\therefore \quad \frac{e_1}{f_1} = \frac{n_2\lambda_2\,s - m^2_2\,(r_1r_2 + r_1r_3 + r_2r_3)}{\lambda^2_2\,s + m^2_2r_1r_2r_3}$$

$$= -\frac{8r_1r_2r_3r_4\,s(r_2r_3r_4 + r_1r_3r_4 + r_1r_2r_4 + r_1r_2r_3)}{16\,r^2_1r^2_2r^2_3r^2_4\,s}$$

$$= -\frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) \quad ......(36)$$

And therefore we get the same result as in (27).

The reciprocal of the sixth radius may be found in a similar way.

Let the four given spheres be equal, or $r_1 = r_2 = r_3 = r_4$; then

$$\mathrm{R} = \frac{r_1}{\sqrt{6}+2} \quad ....(36)$$

$$\mathrm{R}_1 = \frac{r^1}{\sqrt{6}-2} \quad ....(37)$$

$$\because \quad \mathrm{RR}_1 = \frac{1}{2}r^2_1 \quad ....(38)$$

In this case, $n_2 = \alpha\,\beta_1\,\delta_2 - (\alpha_2\,\beta_1 - \alpha_1\beta_2)\delta - \alpha\beta_2\delta_1 = 0$, since $\delta = r_1 - r_2 = 0$, $\delta_1 = r_1 - r_3 = 0$, and $\delta_2 = r_1 - r_4 = 0$; hence, (36, 37) readily follow from (14, 18) and (15, 16, 17).

Again, in (1) put $z = 0$, and substitute, in the resulting equation, the values of x, y, derived from (11, 12); then we have

$$\rho^2 = \frac{(p + n\,\rho)^2}{m^2} + \frac{(p_1 + n_1\,\rho)^2}{m^2_1}.$$

In this let $\rho = \rho_1 + r_1$, and solve for $\dfrac{1}{\rho_1}$,

$$\therefore\ \frac{1}{\rho_1} = -\frac{q}{r} \pm \frac{1}{r}\sqrt{q^2 + r\,s} \quad \ldots\ldots\ldots\ldots\ldots\ldots (39),$$

of which,

$$q = m^2_1 n\lambda + m^2 n_1 \lambda_1 - m^2 m^2_1 r_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (40)$$

$$r = m^2_1 \lambda^2 + m^2 \lambda^2_1 - m^2 m^2_1 r^2_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

$$s = m^2 m^2_1 - (m^2_1 n^2 + m^2 n^2_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

ρ_1 is evidently the radius of a circle which is touched externally by three circles in mutual contact, whose radii are r_1, r_2, r_3.

And if ρ_2 be the radius of the circle which is touched by the same three circles *internally*, we shall have, in a similar way,

$$\frac{1}{\rho_2} = \frac{q}{r} \pm \frac{1}{r}\sqrt{q^2 + r\,s} \quad \ldots\ldots\ldots\ldots\ldots (43)$$

Now, since

$$m^2_1\, n\lambda - m^2\, m^2_1\, r_1 = m^2_1 \{ r_1 (r^2_1 - r^2_2) - r_1 (r_1 + r_2)^2 \} = -\,2 r_1\, r_2\, a\, m^2_1,$$

and $m^2_1\, \lambda^2 - m^2 m^2_1 r^2_1 = 0$,

$$\therefore\ \frac{q}{r} = \frac{m^2\, n_1\, \lambda_1 - 2\, r_1\, r_2\, a\, m^2_1}{m^2\, \lambda^2_1} = \frac{m\, n_1\, \lambda_1 - 2\, r_1\, r_2\, m^2_1}{m\, \lambda^2_1}$$

$$= -\,\frac{4\, r_1 r_2 r_3\, a\, (r_1 r_2 + r_1 r_3 + r_2 r_3)}{4\, a\, r^2_1 r^2_2 r^2_3}$$

$$= -\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) \quad \ldots\ldots (44)$$

$\therefore$ (39, 43, 44)

$$\frac{1}{\rho_1} = 2\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) + \frac{1}{\rho_2},$$

$$\text{or } \frac{1}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \quad \ldots\ldots\ldots\ldots (45)$$

This elegant property may be enunciated thus:

Three circles being placed in mutual contact, and two others being described to be touched by them internally and externally; then half the difference of the reciprocals of the fourth and fifth radii will be equal to the sum of the reciprocals of the other three.

ON THE THEORY AND APPLICATION OF LAGRANGE'S METHOD OF MULTIPLIERS.

[*Mr. Rutherford.*]

The direct method of eliminating unknown or variable quantities by means of an adequate number of equations, frequently leads to complicated and inelegant expressions in the course of the elimination. There is a method, however, which is well adapted for effecting the elimination, and of which a brief and simple illustration may be both interesting and useful to the young student. I allude to the *method of multipliers*. Lagrange, in his *Mecanique Analytique*, tome i, p. 74, has employed this method in determining the conditions of equilibrium of a system of bodies. In several other inquiries it is peculiarly applicable, and not only facilitates the elimination, but necessarily gives rise to expressions and classes of equations frequently remarkable both for their simplicity and symmetry.

In questions concerning the maximum or minimum values of functions of two or more variables, the function to be made a maximum or a minimum must be differentiated, as also the equation or equations of condition to which the variables are subjected. To determine the values of the variables, the differentials of the *dependent* ones must be eliminated, and this, in numerous instances, may be elegantly effected by the method of multipliers. The same method may be advantageously employed in inquiries relating to consecutive lines, curves, and surfaces, for eliminating the variable parameter or parameters; and it is hoped that the observations and illustrations which are here presented to the notice of the student, will be found really useful in the acquisition of a principle of so much utility in various mathematical and physical researches.

For the purpose of illustrating the theory of the method of multipliers, let us take the two simultaneous equations of the first degree,

$$P = 0 \ldots\ldots (1) \qquad Q = 0 \ldots\ldots\ldots (2)$$

where P and Q are functions of two or more unknown quantities, x, y, z, etc. Then if we multiply (1) by λ, and add the product to (2), we shall have the equation

$$\lambda P + Q = 0 \ldots\ldots\ldots\ldots (3)$$

Now, since the equations (1) and (2) are, by hypothesis, simultaneous; that is, since those values of the unknown quantities which satisfy the equation $P = 0$, or $Q = 0$, will also satisfy equation (3); for when in (3) we have $Q = 0$, then $\lambda P = 0$, or $P = 0$; and when $P = 0$, then $\lambda P = 0$, and therefore also $Q = 0$. It is therefore evident that equation (3) involves both the proposed equations (1) and (2), and, by subjecting the multiplier λ to certain conditions, we can obtain the values of as many of the unknown quantities from equation (3) as can be deduced from the proposed equations. If the equations (1) and (2) contain only two unknown quantities, their values may be found from (3) with as much facility and elegance as from the original equations.

Again, if we have the three simultaneous equations of the first degree,

$$P = 0 \ldots\ldots (1) \qquad Q = 0 \ldots\ldots (2) \qquad R = 0 \ldots\ldots (3)$$

where P, Q, R are functions of three or more unknown quantities; then

multiplying (1) by λ, (2) by λ_1, and adding the products to (3), we have the equation

$$\lambda P + \lambda_1 Q + R = 0 \ldots\ldots\ldots\ldots (4)$$

which combines all the three equations (1, 2, 3); because it may be written in either of the forms

$$\lambda P + (\lambda_1 Q + R) = 0 \ldots\ldots\ldots\ldots (5)$$

$$\lambda_1 Q + (\lambda P + R) = 0 \ldots\ldots\ldots\ldots (6)$$

and it must therefore be satisfied by those values of the unknown quantities which satisfy either of the systems

$$\begin{array}{cc} P = 0 & Q = 0 \\ R = 0 & R = 0 \end{array}$$

for those values of the unknown quantities which satisfy these systems, have been shown in the preceding case of two simultaneous equations, to satisfy also

$$\lambda P + R = 0 \text{ and } \lambda_1 Q + R = 0;$$

and hence again, by the same case, the two forms (5) and (6), each of which is identical with (4), must be satisfied by those same values. The equation (4) is, therefore, the conjoint equation of the proposed system of equations (1, 2, 3); and thus the operation of combining, by the method of multipliers, any number of equations, may be carried out to any extent. If each of the equations (1, 2, 3), contains only three unknown quantities, their values may be found from equation (4) alone, by subjecting the multipliers λ and λ_1 to certain conditions. The conditions to which these multipliers are subjected will furnish a sufficient number of equations by which their values may be *determined*, and this, we think, is one reason amongst many others, for discarding the phrase *indeterminate multipliers*, usually employed, and replacing it by the more appropriate one of *conditional multipliers*.

To render the method of conditional multipliers familiar to the student, we shall apply it to the resolution of some very easy examples in different branches of science.

1. Find the values of x and y from the simultaneous equations

$$a x + b y + c = 0,$$

$$a_1 x + b_1 y + c_1 = 0.$$

Multiply the former by λ, and add the product to the second; then we get

$$(\lambda a + a_1) x + (\lambda b + b_1) y + (\lambda c + c_1) = 0,$$

and to determine from this equation alone the values of both the unknowns, we must make the coefficients of these unknowns successively equal to zero; hence, if

$$\lambda a + a_1 = 0; \text{ then } \lambda = -\frac{a_1}{a}, \text{ and } y = -\frac{\lambda c + c_1}{\lambda b + b_1} = \frac{a_1 c - c_1 a}{a b_1 - b a_1},$$

$$\lambda b + b_1 = 0; \text{ then } \lambda = -\frac{b_1}{b}, \text{ and } x = -\frac{\lambda c + c_1}{\lambda a + a_1} = \frac{b c_1 - c b_1}{a b_1 - b a_1}.$$

As a numerical example, take the equations

$$7x + 2y - 30 = 0,$$

$$5x + 3y - 34 = 0.$$

Multiplying the first by λ, and adding the product to the second, we get

$$(7\lambda + 5)x + (2\lambda + 3)y - (30\lambda + 34) = 0.$$

$$\text{If } 2\lambda + 3 = 0;\ \text{then } \lambda = -\frac{3}{2},\ \text{and } x = \frac{30\lambda + 34}{7\lambda + 5} = 2,$$

$$7\lambda + 5 = 0;\ \text{then } \lambda = -\frac{5}{7},\ \text{and } y = \frac{30\lambda + 34}{2\lambda + 3} = 8.$$

2. Find the values of x, y, z from the three equations

$$ax + by + cz + d = 0,$$
$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0.$$

Multiply the first by λ, the second by λ_1, and add the products to the third; then

$$(\lambda a + \lambda_1 a_1 + a_2)x + (\lambda b + \lambda_1 b_1 + b_2)y + (\lambda c + \lambda_1 c_1 + c_2)z + (\lambda d + \lambda_1 d_1 + d_2) = 0;$$

and since this equation is the conjoint equation of the three given ones, we must subject the multipliers to the conditions that the coefficients of any two of the unknown quantities shall be zero; hence, to find the value of x, we must have

$$\lambda b + \lambda_1 b_1 + b_2 = 0,$$
$$\lambda c + \lambda_1 c_1 + c_2 = 0;$$
$$\therefore\ x = -\frac{\lambda d + \lambda_1 d_1 + d_2}{\lambda a + \lambda_1 a_1 + a_2}.$$

From the two former of these we get, by the first example, the values of the multipliers λ and λ_1, and thence, by substitution, the value of x.

It is unnecessary to exhibit in full the values of the unknown quantities, as they are given in most elementary works on Algebra, our object being simply to indicate the method of obtaining them by using conditional multipliers. The student will readily apply the principle to numerical examples.

3. Inscribe the greatest rectangular parallelopiped in a given ellipsoid.

Let a, b, c be the principal semi-axes of the ellipsoid, and x, y, z the semi-edges of the parallelopiped; then the volume of the parallelopiped is $8xyz$, and if this be a maximum, its logarithm must be a maximum; hence, by this condition, and the equation of the ellipsoid, we have

$$u = \log_e x + \log_e y + \log_e z = \text{maximum} \ldots\ldots (1)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots\ldots (2)$$

By the principles of maxima and minima, the differential of (1) must be zero; hence, differentiating both equations, we get

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \ldots\ldots\ldots (3)$$

$$\frac{xdx}{a^2} + \frac{ydy}{b^2} + \frac{zdz}{c^2} = 0 \ldots\ldots\ldots (4)$$

Multiply (3) by λ, and add the product to (4); then we have

$$\left(\frac{\lambda}{x} + \frac{x}{a^2}\right)dx + \left(\frac{\lambda}{y} + \frac{y}{b^2}\right)dy + \left(\frac{\lambda}{z} + \frac{z}{c^2}\right)dz = 0 \ldots (5)$$

But by the principle of the method of multipliers we must make the

coefficients of dx and dy each equal to zero; hence also, the coefficient of dz must necessarily be equal to zero; consequently we have

$$\frac{\lambda}{x}+\frac{x}{a^2}=0\ldots\ldots\ldots(6)$$

$$\frac{\lambda}{y}+\frac{y}{b^2}=0\ldots\ldots\ldots(7)$$

$$\frac{\lambda}{z}+\frac{z}{c^2}=0\ldots\ldots\ldots(8)$$

Then multiplying (6), (7), (8) respectively by x, y, z, and taking the sum of the products, we obtain, by the aid of equation (2),

$$3\lambda+1=0 \qquad \therefore \lambda=-\frac{1}{3}.$$

Substituting this value of λ, in (6, 7, 8), we get at once

$$x=\frac{a}{\sqrt{3}},\ y=\frac{b}{\sqrt{3}},\ z=\frac{c}{\sqrt{3}};$$

and therefore $u=8xyz=\dfrac{8abc}{3\sqrt{3}}=$ volume of maximum parallelopiped.

4. Find the values of x, y, z, when x^4yz^2 is a maximum, and subject to the condition

$$x^2+2y^3+z^4=a.$$

Taking the logarithm of the expression to be made a maximum, we get

$$u=4\log_e x+\log_e y+2\log_e z=\text{maximum}\ldots\ldots(1)$$

$$x^2+2y^3+z^4=a\ \ldots\ldots\ldots\ldots(2)$$

Differentiating these equations, we have

$$\frac{4dx}{x}+\frac{dy}{y}+\frac{2dz}{z}=0\ldots\ldots\ldots\ldots(3)$$

$$xdx+3y^2dy+2z^3dz=0\ldots\ldots\ldots\ldots(4)$$

Multiplying equation (3) by λ, and adding the product to (4), gives

$$\left(\frac{4\lambda}{x}+x\right)dx+\left(\frac{\lambda}{y}+3y^2\right)dy+\left(\frac{2\lambda}{z}+2z^3\right)dz=0;$$

therefore we must have

$$\frac{4\lambda}{x}+x=0\ldots(5);\quad \frac{\lambda}{y}+3y^2=0\ldots(6);\quad \frac{\lambda}{z}+z^3=0\ldots(7);$$

Multiply (5), (6), (7) by x, $\frac{2}{3}y$*, and z respectively, and take the sum of the products; then

$$4\lambda+\lambda+\tfrac{2}{3}\lambda+x^2+2y^3+z^4=0;\ \text{or,}$$

$$\tfrac{17}{3}\lambda+a=0 \quad \therefore \lambda=-\tfrac{3}{17}a;$$

* It may not be unimportant to remind the student that the quantities by which the equations involving the conditional multiplier are to be multiplied, must be determined by the proposed equations of condition. Thus if we multiply (5) by x, we shall have one term in the product equal to one term of (2); but since the second term of eq. (2) is $2y^3$, we must necessarily multiply (6) by $\frac{2}{3}y$ in order that one term in this product may agree with the corresponding term of (2). In the same way the proper multiplier in each case, may be found.

and substituting this value of λ in (5), (6), (7), we obtain

$$x^2 = \frac{12}{17}a, \quad y^3 = \frac{1}{17}a, \quad z^4 = \frac{1}{17}a.$$

5. If a, b, c, be the segments of the axes of co-ordinates cut off by a plane, and be subject to the condition $a\,b\,c = m^3$; what is the equation of the surface which is always touched by the plane?

The equation of the plane, and the equation of condition are respectively

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \ldots\ldots (1)$$

$$abc = m^3 \ldots. (2)$$

Taking the logarithm of equation (2) we have

$$\log_e a + \log_e b + \log_e c = 3 \log_e m \ldots\ldots (3);$$

and differentiating both (1) and (3), with regard to the variable segments a, b, c, we have

$$\frac{xda}{a^2} + \frac{ydb}{b^2} + \frac{zdc}{c^2} = 0 \ldots\ldots (4)$$

$$\frac{da}{a} + \frac{db}{b} + \frac{dc}{c} = 0 \ldots\ldots (5)$$

Multiply (4) by λ, add the product to (5), and then equate the co-efficients of the differentials da, db, dc, separately to zero; this gives

$$\frac{\lambda x}{a^2} + \frac{1}{a} = 0 \ldots\ldots (6)$$

$$\frac{\lambda y}{b^2} + \frac{1}{b} = 0 \ldots\ldots (7)$$

$$\frac{\lambda z}{c^2} + \frac{1}{c} = 0 \ldots\ldots (8)$$

Multiplying (6), (7), (8), by a, b, c, respectively, and adding the results, gives, by the first equation,

$$\lambda\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) + 3 = 0, \text{ or } \lambda + 3 = 0 \ldots (9)$$

Put the value of λ from (9), in (6), (7), (8), and we have at once

$$a = 3x, \quad b = 3y, \quad c = 3z;$$

and these values of a, b, c, being substituted in the equation of condition (2) give, finally,

$$27xyz = m^3, \text{ or } xyz = \left(\frac{m}{3}\right)^3;$$

which is the equation of the surface constantly touched by the plane cutting off from the axes the segments a, b, c, whose product is always a given quantity.

6. To determine the magnitude of the principal axes of central curves.

Let the equation of the curve be

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots (1)$$

and let the angle of ordination of the oblique axes be denoted by θ.

Let α, β, be the co-ordinates of the centre, and r any variable radius; then

$$r^2 = (y-\beta)^2 + 2(x-\alpha)(y-\beta)\cos\theta + (x-\alpha)^2 = \text{max.} \ldots (2)$$

Removing the origin of co-ordinates to the centre we get for the transformed equation

$$a(y-\beta)^2 + b(x-\alpha)(y-\beta) + c(x-\alpha)^2 + f_1 = 0 \ldots\ldots (3)$$

$$\text{where } \alpha = \frac{2ae - bd}{b^2 - 4ac}, \ \beta = \frac{2cd - be}{b^2 - 4ac};$$

$$\text{and } f_1 = a\beta^2 + b\alpha\beta + c\alpha^2 + d\beta + e\alpha + f$$

$$= \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f = \frac{d\beta + e\alpha}{2} + f.$$

Now by differentiating (2) and (3), and recollecting that (2) is to be max., we have these two equations,

$$\{(y-\beta) + (x-\alpha)\cos\theta\}dy + \{(x-\alpha) + (y-\beta)\cos\theta\}dx = 0 \ldots (4)$$

$$\{2a(y-\beta) + b(x-\alpha)\}dy + \{2c(x-\alpha) + b(y-\beta)\}dx = 0 \ldots (5)$$

Multiply (4) by the conditional multiplier λ, and add the product to (5); then equating to zero the coefficients of the differentials dy and dx, we get

$$\lambda\{(y-\beta) + (x-\alpha)\cos\theta\} + 2a(y-\beta) + b(x-\alpha) = 0 \ldots (6)$$

$$\lambda\{(x-\alpha) + (y-\beta)\cos\theta\} + 2c(x-\alpha) + b(y-\beta) = 0 \ldots (7)$$

Multiply (6) by $(y-\beta)$ and (7) by $(x-\alpha)$; then taking the sum of the products, and simplifying the resulting equation by means of (2) and (3), we obtain

$$\lambda r^2 - 2f_1 = 0; \text{ or } \lambda = \frac{2f_1}{r^2} \ldots\ldots (8)$$

Substituting in (6) and (7) for λ its value as obtained in (8) we get

$$2(ar^2 + f_1)(y-\beta) = -(br^2 + 2f_1\cos\theta)(x-\alpha) \ldots\ldots (9)$$

$$2(cr^2 + f_1)(x-\alpha) = -(br^2 + 2f_1\cos\theta)(y-\beta) \ldots\ldots (10)$$

Taking the product of (9) and (10) we have finally

$$4(ar^2 + f_1)(cr^2 + f_1) = (br^2 + 2f_1\cos\theta)^2;$$

which, expanded and arranged, with respect to r, gives

$$(b^2 - 4ac)r^4 - 4f_1(a - b\cos\theta + c)r^2 = 4f_1^2\sin^2\theta \ldots (11)$$

$$\therefore r^2 = \frac{2f_1\{a - b\cos\theta + c \pm \sqrt{(a - b\cos\theta + c)^2 + (b^2 - 4ac)\sin^2\theta}\}}{b^2 - 4ac} \ldots (12)$$

and if we assume

$$Q = a - b\cos\theta + c,$$

$$R^2 = (a - b\cos\theta + c)^2 + (b^2 - 4ac)\sin^2\theta,$$

$$= (a-c)^2 + (b - 2a\cos\theta)(b - 2c\cos\theta);$$

then we get

$$r^2 = \frac{2f_1(Q \pm R)}{b^2 - 4ac}, \text{ and } r = \pm\left\{\frac{2f_1(Q \pm R)}{b^2 - 4ac}\right\}^{\frac{1}{2}},$$

an expression involving the magnitudes of the semi-axes, major and minor,

the $+$ giving the semi-major axis, and the $-$ giving the semi-minor axis.* As an illustration of this take the following example:

Find the magnitude of the principal axes of the ellipse whose equation referred to rectangular axes, is

$$5y^2 + 2xy + 5x^2 + 2y - 2x - 1\cdot 5 = 0.$$

Here $Q = a + c = 10$, and $R^2 = (a - c)^2 + b^2 = b^2 = 4$; also

$$f_1 = \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f = -\tfrac{1}{2} - \tfrac{3}{2} = -2,$$

$$\therefore r_1 = \left\{ \frac{2f_1(Q + R)}{b^2 - 4ac} \right\}^{\frac{1}{2}} = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2} = \text{semi-major axis},$$

$$r_2 = \left\{ \frac{2f_1(Q - R)}{b^2 - 4ac} \right\}^{\frac{1}{2}} = \sqrt{\tfrac{1}{3}} = \tfrac{1}{3}\sqrt{3} = \text{semi-minor axis}.$$

ON CONJUGATE DIAMETERS OF LINES AND SURFACES OF THE SECOND ORDER.

[*Mr. Fenwick.*]

Two systems of points, which are so related, that to each *point* of the one system there is a corresponding *line* of the other system, and to each point of the second system a corresponding line of the first system, are called a *conjugate system*. The point and the line corresponding to it, are called in reference to each other, *conjugate polars*.

Let, then, $x_1\, y_1$ be the co-ordinates of a point in the one system, and xy, those of a point in the other, referred to any axes whatever: then if to a given point $(x_1\, y_1)$ the line whose equation is

$$my + nx + p = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

is to correspond, it will be obvious from the specified relation, that m, n and p, must be functions of x_1, and y_1, and of the first degree. Hence (1) becomes in general

$$(ay_1 + bx_1 + c)\, y + (a_1y_1 + b_1x_1 + c_1)x + a_2y_1 + b_2x_1 + c_2 = 0 \ldots . (2)$$

We can thence find the co-ordinates (x_1y_1) of the pole of a given polar line, by considering the equation of this polar line and (2) *identical*, and equating the co-efficients of the like powers of the variables.

* The same resulting formula for determining the values of the semi-axes of central curves is readily deducible from the equations (7) or (8) at page 17 of the present number. For if, in accordance with the concluding remarks in the page referred to, we equate the expression under the radical to zero, we have

$$(br^2 + 2f\cos a)^2 = 4\,(ar^2 + f)\,(cr^2 + f);$$

$$\therefore (b^2 - 4ac)r^4 - 4f\,(a - b\cos a + c)r^2 = 4f^2\sin^2 a.$$

Then resolving for r, and employing the above notation, we get

$$r = \left\{ \frac{2f\,(Q \pm R}{b^2 - 4ac} \right\}^{\frac{1}{2}}.$$

Again, the equation of a line

$$y = px + q \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

in which, p is *constant*, and q *variable*, represents all lines which are parallel to the same line, and the pole (x_1y_1) of one of these will be determined by equating the coefficients of the variables in (2, 3). Hence, we have

$$a_1y_1 + b_1x_1 + c_1 + p\,(ay_1 + bx_1 + c) = 0 \ldots\ldots(4)$$

$$\text{and } a_2y_1 + b_2x_1 + c_2 + q\,(ay_1 + bx_1 + c) = 0 \ldots\ldots(5)$$

Now, since p is *constant*, (4) is the equation of a *determinate straight line*, and hence the poles of all parallel lines lie in the same straight line.

This line, which contains the poles of all the lines which are parallel to the same line, is named the diameter of the system which is conjugate to that line.

The following investigations will serve for illustration.

1. If referred to any system of rectilinear co-ordinates there be the conic section $ay^2 + bxy + cx^2 + dy + ex + f = 0$, and the two straight lines $y = px + q$, and $y = p_1x + q_1$: then, if $2app_1 + b\,(p + p_1) + 2c = 0$, the two lines will be conjugate to each other in respect of the conic section (*Lady's and Gentleman's Diary*, for 1843).

Now, in Davies's *Hutton*, vol. ii. p. 305, it is shewn, that if there be the conic section $ay^2 + bxy + cx^2 + dy + ex + f = 0$, the point (x_1y_1), and the line

$$(2\,ay_1 + bx_1 + d\,)\,y + (2\,cx_1 + by_1 + e)\,x + dy_1 + ex_1 + 2f = 0 \,..(6);$$

then the point (x_1y_1) and the line (6) have the relation of conjugate polars. Instead therefore of (2), we shall employ (6), in connexion with curves of the second degree.

In order, now, to find the equation of the line conjugate to $y = px + q$, let us consider p constant, and q variable; then, equating the coefficient of x in this line with that of x in (6), we have

$$2cx_1 + by_1 + e + p\,(2ay_1 + bx_1 + d\,) = 0,$$

or, arranging according to y_1 and x_1, and dropping the subscribed numerals,

$$(2ap + b)\,y + (2c + bp)\,x + e + pd = 0 \ldots\ldots(7)$$

Hence, that the line whose equation is $y = p_1x + q_1$, and (7), may be parallel, we must have the relation

$$\frac{2c + bp}{2ap + b} = -\,p_1,$$

$$\text{or, } 2app_1 + b\,(p + p_1) + 2c = 0,$$

which is the enunciated criterion.

2. An infinite number of curves of the second degree can be made to pass through four given points in the same plane; in each of these curves there is drawn the diameters of the chords which are parallel to a given line, having for its equation $y = nx$. All these diameters intersect in the same point.

In this case, we must find the equation of the line conjugate to that whose equation is

$$y = nx + p \ldots\ldots\ldots\ldots(8)$$

and in which, n is constant and p variable.

Hence, equating the coefficients of x in (6, 8), we have

$$2cx_1 + by_1 + e + n\,(2ay_1 + bx_1 + d) = 0,$$

$$\text{or, } b\,(y_1 + nx_1) + 2any_1 + 2cx_1 + e + dn = 0 \ldots\ldots\ldots (9)$$

Now, as the curve passes through *four points only*, one coefficient (b) of the equation of the curve remains *indeterminate*; hence, (9) becomes, in consequence of the *arbitrary quantity*, b,

$$y_1 + nx_1 = 0, \text{ and } 2any_1 + 2cx_1 + e + dn = 0.$$

The values of x_1 and y_1 deduced from these two equations, will be the co-ordinates of a point which, at the same time, belongs to all the diameters. It is proved, therefore, that all the diameters pass through the same point.

We proceed to find a conjugate diameter in space.

The relation, in this case, must exist between a point and a plane.

If, then, to a given point $(x_1\ y_1\ z_1)$, the plane whose equation is

$$mz + ny + px + q = 0 \ldots\ldots\ldots\ldots\ldots (10)$$

is to correspond, m, n, p, and q must be functions of x_1, y_1, and z_1, and of the first degree. Hence (10) becomes

$$(az_1 + by_1 + cx_1 + d)\,z + (a_1z_1 + b_1y_1 + c_1x_1 + d_1)\,y + (a_2z_1 + b_2y_1 + c_2x_1 + d_2)\,x$$
$$+ a_3z_1 + b_3y_1 + c_3x_1 + d_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Let now, in the equation

$$z = n_1y + p_1x + q_1 \ldots\ldots\ldots\ldots (12)$$

n_1 and p_1 be constant, and q_1 variable; then (12) will represent all planes which are parallel to the same plane, and the pole of one of them is obtained from the following equations, derived from (11, 12): viz.

$$a_1z_1 + b_1y_1 + c_1x_1 + d_1 + n_1(az_1 + by_1 + cx_1 + d) = 0 \,..\, (13)$$
$$a_2z_1 + b_2y_1 + c_2x_1 + d_2 + p_1(az_1 + by_1 + cx_1 + d) = 0 \,..\, (14)$$
$$a_3z_1 + b_3y_1 + c_3x_1 + d_3 + q_1(az_1 + by_1 + cx_1 + d) = 0 \,..\, (15)$$

Hence, since n_1 and p_1 are constant, (13) and (14) are the equations of a *determinate straight line*, and therefore the poles of all parallel planes are in the same straight line.

This line which contains the poles of all the planes which are parallel to the same plane, is named the diameter of the system which is conjugate to that plane.

The following theorem is readily deduced:

An infinite number of surfaces of the second order may pass through eight given points in space. Let sections be cut parallel to a given plane; then will the diameters which are conjugate to these sections all meet in the same point (*Lady's and Gentleman's Diary*, for 1843).

For, if (12) in its varying positions, cut a surface whose equation involves an *arbitrary quantity*, this arbitrary quantity will enter into (13) or (14), representing the diameter conjugate to the plane to which (12) is always parallel, and hence we shall have from (13, 14) three equations of the form

$$f(z_1, y_1, x_1) = 0\,;\ f_1(z_1, y_1, x_1) = 0\,;\ f_2(z_1, y_1, x_1) = 0\,;$$

in which z_1, y_1, and x_1, are of the first degree.

The values of $x_1\ y_1\ z_1$, deduced from these three equations, will be the co-ordinates of a point which, at the same time, belongs to all the diameters.
· the truth of the theorem.

MATHEMATICAL EXERCISES.

1.—*Mr. R. H. Wright, London.*

If a and b be respectively the semi-axes major and minor of an ellipse, and a body be projected from one extremity of its axis major, so as to exactly shoot down an inclined plane, meeting the extremity of the axis minor and the other extremity of the axis major; show that, if e be the angle of projection, and $\cos e_1$ the eccentricity of the ellipse,

$$\tan e = 3 \sin e_1.$$

2.—*By a Mathematician.*

Can the expression $(-1)^{h^{\frac{m}{n}}}$ be real for any values of h, m, and n? and if so, discriminate the cases.

3.—*Mr. Fenwick.*

If the sides and angles of the triangle formed by joining the centres of the escribed circles of a given triangle be denoted by a_1, b_1, c_1, and A_1, B_1, C_1, respectively; and its mass by M_1: then the moment of inertia of the original triangle, with respect to a line perpendicular to its plane through its centre of gravity, is

$$\frac{1}{18} M_1 \cos A_1 \cos B_1 \cos C_1 (a^2_1 \cos^2 A_1 + b^2_1 \cos^2 B_1 + c^2_1 \cos^2 C_1).$$

4.—*Mr. Rutherford.*

If α, β are the co-ordinates of the centre of a circle which cuts an ellipse in four points, and the equations of the ellipse and circle be

$$a^2y^2 + b^2x^2 = a^2b^2, \text{ and } (y-\beta)^2 + (x-\alpha)^2 = r^2,$$

then the product of the distances of the four points of intersection from the major axis is

$$\frac{b^4\{(a^2 + \alpha^2 + \beta^2 - r^2)^2 - 4a^2\alpha^2\}}{(a^2-b^2)^2}.$$

5.—*Mr. Fenwick.*

From the angles A, B, C, of a triangle, draw lines through any point P, to meet the opposite sides in D, E, F; and join DE, EF, FD, meeting CF, AD, BE, in f, d, e. Having joined fd, de, ef, draw lines from A, B, C, and D, E, F, to bisect EF, DF, DE, and ef, df, de, respectively; the former meeting in M, and the latter in N: then will M, N, P range in a straight line.

6.—*Mr. Davies.*

What are the properties in reference to surfaces of the second order, which correspond to those of Pascal and Brianchon, in reference to lines of the second order?

7.—*Mr. Rutherford.*

Find the magnitude and position of a circle which touches three given circles in mutual contact described on the surface of a given sphere.

[*Puissant, Propositions de Géométrie*, p. 380.]

[We beg to call the attention of our Mathematical friends to the views which we have explained in the prospectus, relative to this department of our work: and we most earnestly urge upon them the deduction of as many consequences of the results themselves as well as of the truths employed in their investigation, as they can obtain—being convinced that no other mode of study so much *tends to increase* the power and improve the taste of *the young student.*]

ON THE ALGEBRAICAL ANALYSIS OF PORISMS.

[*Mr. Davies.*]

This celebrated enigma of antiquity was resolved by Simson only during the last century; and even that solution has been strenuously contested by a no less distinguished geometer than Chasles.

Like our knowledge of almost every thing that is valuable in the Greek geometry,—the Elements of Euclid, the Conics of Apollonius, and a few of the writings of Archimedes, excepted,—our knowledge of the subject of *Porisms*, the most refined of all the ancient speculations, is derived from a very brief and mutilated fragment preserved in the preface to the seventh book of the mathematical collections of Pappus. Dr. Simson was the first to unravel the meaning of that obscure passage, though it had been frequently attempted before his time. The only attempt, however, which approximated to success was that of Fermat: and, even though he certainly did discover a few actual porisms of one particular class, he altogether failed to discover the origin, peculiar character, purposes, or mode of investigation of this species of proposition. Nor, indeed, could we expect his success to be greater; since he founded his conjectures on a definition of the porism, which Pappus expressly says was only an *accidental* circumstance belonging to *some* of these propositions. It is strictly true, as Playfair has remarked, that "the complete restoration of the Porisms was necessary, to prove that Fermat had even approximated to the truth." In a very great degree, however, the origin, character, and purpose of the porism has been satisfactorily and completely developed by Simson; and the subject has been copiously and most ably illustrated by Playfair. At the same time, as far as the analysis adapted to this class of propositions is considered, (if indeed, the ancients had any analysis* adapted to them) there is some reason to sus-

* The earliest writer mentioned by Pappus upon geometrical analysis is Euclid, and it is expressly stated that this instrument of research was principally applied to *problems*. It is a general opinion, that before the time of Euclid, no branch of geometrical inquiry had taken a strict and definite form; yet when we consider the variety and extent of the subjects upon which Euclid treated, and the extraordinary degree of clearness and precision by which they are all characterised, this view appears to be highly improbable. That preceding works, of perhaps great merit, upon almost every branch of geometry existed before Euclid's time, but which fell into neglect in consequence of the superiority of those of Euclid and Apollonius, and have, hence, been altogether lost, is much more likely.

That the *data* of Euclid was written for problematic investigation, its internal evidence,—even the very form,—leaves no doubt; but whether he had extended the method to speculative truths, we have no evidence of any kind to adduce, on one side or the other. The extremely close analogy between the speculative analysis and the method *reductio ad absurdum*, so extensively employed in the Elements, renders it almost impossible that he should not have discovered the application of the analytic method to theoretic investigations, even had no such application been made before his time. In fact, the two methods differ only in this:—that in *analysis* we suppose the theorem true, and in the *reductio*, we suppose it false; whilst, in both cases, the chains of consequences deducible from these hypotheses are pursued till they end in a conclusion known, respectively, to be true or to

pect that Simson has been less successful. Still, even if he should not be found to have restored the true analysis of the ancients, he has invented one fully adequate to the purpose, and which can always be applied to the fulfilment of its object.

My intention originally, was to give a tolerably copious account of the labours of those writers who have treated on this subject, and to discuss at some length several important questions respecting it, which have not yet received all the attention they deserve. I was, also, especially desirous of examining the objections so forcibly stated by Chasles to the English view of the Greek porism. Upon looking, however, to my notes, I find that the space which can be allotted to this subject in the present number of the Mathematician, would be inadequate to the purpose, without abandoning the main object of this paper,—an illustration of the algebraic analysis of porisms. Perhaps, in the end, to defer these discussions may be advantageous with respect to the simplification and perspicuity of the entire system of reasoning.

Euclid's porisms, as we learn from Pappus, related entirely to straight lines and circles: but, generally speaking, any other defined loci might be made the subject of porismatic inquiries. There is every evidence, too, that these propositions related to indeterminate magnitudes: and that they also included both a theorem and a problem in each particular instance. I shall, hence, propose the following definition of the porism, which will meet the most general views yet offered of the nature of these propositions, and which is in strict accordance with the general enunciation of all porisms.

Let a problem be such as to require for the complete determination of some entity,* the enunciation of n data: then a porism states that if $n-m$ of these data be actually chosen, the remaining m entities may be so determined as to render the problem indeterminate; and requires that we actually determine these m entities in relation to the data and to each other.

As we are about to employ the algebraical calculus in porisms almost wholly geometrical in this paper, it may be necessary to remark that:—

(1.) A point is not, generally,† the *quæsitum* of the fundamental problem: for a point becoming indeterminate only produces a locus, and it is clear from the definition censured by Pappus, that a porism was something different from a local theorem. Want of attention to this circumstance has led many distinguished writers into error:—as, for instance, Lord Brougham, *Phil. Trans.* 1798, prop. viii, who has given a local theorem as a porism (and the wrong locus, too, accidentally assigned,) and Mr. Babbage, *Ed. Trans.*

be false. Is it not probable, therefore, that Euclid himself made this extension of the method of analysis? And, if so, is it not possible that he might also have invented an analysis adapted to porisms? The remark of Pappus, respecting the chief destination of analysis in the ancient geometry, is the most powerful reply, in a negative shape, that, it appears to me, can be given: but at a future time I shall have occasion to recur to this question more in detail, and will consequently add nothing further on the subject here.

* The word "*thing*" would be a more obvious term as a translation of Simson's "*res;*" but I have preferred to borrow a term familiar to writers on the philosophy of the human mind, and which corresponds to the word "*being,*" used substantively. These geometrical or analytical entities may be magnitudes, ratios, positions, specified figures, or abstract symbols; in short, whatever may become the subjects of mathematical inquiry.

† The particular case by which the word "never" would be rendered incorrect, will be discussed on a future occasion.

vol. ix., almost every one of whose professed porisms is essentially a local theorem.

(2.) With the ancients, the indeterminate entities were straight lines or circles, and the porismatic* entities were any whatever, as lines, angles, circles, *etc.*

Now a line requires for its complete determination two conditions, and a circle three; and, hence, if of the two or three conditions respectively required, one be wanting, that line, or that circle will become indeterminate. In the solution of a problem, if all the conditions be given, and the construction to which we are led shall be such that *all the lines* or *all the circles* that fulfil certain of the conditions, *must* pass through a determinate point, then there will be a porismatic case of that problem. For, if in this case, we suppose that $(n - 1)$ conditions have been used, and that the remaining datum be a point; then as the n^{th} datum is a point, it may have the same situation as the fore-named constructive point, and the problem becomes indeterminate, that constructive point becoming what we have called the porismatic point. This is indeed a simple case; yet it is that which most commonly occurs. It will, however, sometimes happen that there are two or three, or even more, porismatic points, lines, circles or angles: but in all cases the general principle is the same, *mutatis mutandis*, in its application to every variety of instance. More specific details must be deferred till a future time; as sufficient has been stated for our present purpose.

(3.) In the application of algebra to this class of geometrical researches, I have found it in general most convenient to employ the co-ordinate method; though in a very few cases, the ordinary forms of determinate problems have given the most simple analysis. Whether Playfair had in view any other than this latter method in the paper which he promised (but never published) on the treatment of porisms by algebra cannot be very easily decided. The probability is that, from the little which was known in this country of the co-ordinate method, he had no intention of applying that method. I am led to think that this idea (the employment of co-ordinates) was first proposed by myself in a paper which I sent to Professor Leybourn in 1824, but which was not published in his Mathematical Repository; whilst a few years later a paper in which the method is employed was published in the same work. Those who know the details of management of the Repository, will understand both circumstances; whilst to those who do not, it is sufficient to say:— that I was then unknown as an author, that Professor Leybourn always put the papers he received into the hands of his colleagues for a judgment on them, and that the article in question was written by one of those colleagues.

* The indiscriminate use of the word *given*, in the sense of a *datum*, and that of a *thing which is determinable*, has been often censured, both by English and foreign writers. Whether, however, when used as commonly done in the analysis of a problem, this objection be well founded, I have much doubt: for there is, in all cases, a reference made to some authority (or at least implied when the dependence is obvious or generally known) in which the dependent entity is actually obtained from the data. In the case of porisms, on the contrary, the *determinateness* of these entities (or their *determinability*, if such a term may be allowed) is all that is alledged; and hence the use of the word *given*, to express these, is not only liable to create misconception, but is really a perversion of the term to a sense quite the reverse of its usual one. A writer, to whom hereafter I shall have occasion to refer more particularly, proposes to print the word *given*, when used in the sense of "not given, but which may be found," in italics, reserving the ordinary type for expressing the data and its analytic consequences. It ars to me however, that to call these *porismatic points*, *lines*, *etc.*, will satisfy every ıd which precision can make, besides possessing some other advantages.

(4.) The application of algebra, as far as method is concerned, to the investigation of porisms, is tolerably obvious. The data are to be expressed in the notation usual in such cases, viz. the co-ordinates of a point, the equation of a line or circle, *etc.*, in which the constants are expressed by their common symbols: the porismatic parts by suitable notation, their constants being considered as *unknown:* and the indeterminate quantities of the porism must have their coefficients expressed in terms of the data and unknowns, according to the stated relation of the given and porismatic parts in connection with the indeterminates. Then in accordance with the principles of *conditional coefficients*, as laid down in the first paper of the present number, these several coefficients must be simultaneously made zero; and from the equations thus obtained, the value of the unknown or porismatic quantities must be deduced. If these results be real and consistent, the porism is established; and if imaginary or inconsistent the alleged porism is erroneously stated.

(5.) As in the ordinary application of algebra to geometry, so in this, it is found that cases of the proposition are *suggested* by the result, which would otherwise escape our notice. The proposition, too, may by this method be treated and even enunciated, with less restriction than when treated geometrically: for it will be seen that frequent conditions which geometry (in order that its powers may extend to the porism) requires to be given in the hypothesis, are here shewn to be *inevitable* conditions of the truth of the porism. Geometry proves that under the specified conditions the porism is true: algebra, besides this, shews that it can be true in no other. It would, therefore, seem desirable that even where we propose to investigate the porism geometrically, we should *also* investigate it algebraically.

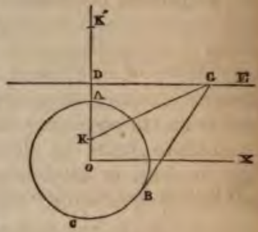
(6.) The equations given in the final solution of a problem by algebra furnish a complete answer to the problem, and hold a corresponding place to the construction of a geometrical problem. It would hence be mingling together two methods which are essentially distinct from each other, were we to "construct the resulting equations:" though in most cases, if not in all, it would be very easy to deduce geometrical constructions from the final results of our analysis of the porisms: and it would generally be found that the simplest geometrical constructions possible of the several porisms are those which result from analysis of the present kind; thus justifying the statement of Playfair, that "in the algebraical analysis a method of investigating these propositions will present itself which is more simple and direct than any other." The *resulting constructions* are, however, principally on the ground of expense, omitted in the following pages: though the subject may, possibly, be resumed hereafter, should the public approbation of the investigations now given, warrant the application of our funds to the engravings which will be required.

The following examples, with the remarks occasionally appended to them will afford sufficient illustration both of the general principle and its applications. These examples are almost entirely chosen from such as have been treated geometrically by preceding writers; and some of the places where they occur are specified after the enunciations.

PORISM I.

A circle, ABC, *being given in position, and also a straight line* DE, *which does not cut the circle, a point* K *may be found, such that if* G *be any point whatever in* DE, *the straight line* GK *shall be equal to the straight line drawn from* G *touching the circle* ABC.

[Playfair, Ed. Trans. ii. prop. 1; Simson, de porism., prop. 66.]



Through the centre of the circle, O, draw OX parallel and OD perpendicular to the line DE, and take these as axes of co-ordinates. Put the radius OA $= r$, OD $= a$; and denote the arbitrary point G in DE by $x'\,a$, and the porismatic point K by $\alpha\,\beta$. Then*

$$GB^2 = x'^2 + a^2 - r^2, \text{ and}$$
$$GK^2 = (x' - \alpha)^2 + (a - \beta)^2.$$

Equate these according to the porism: then we get

$$2\alpha x' - \alpha^2 + 2a\beta - \beta^2 - r^2 = 0.$$

Now equating the coefficient of the arbitrary x' to zero, we have

$$2\alpha = 0, \text{ or } \alpha = 0 \ldots\ldots(1)$$
$$\beta = a \pm \sqrt{a^2 - r^2} \ldots.(2)$$

From (1) we learn that the point K is in the perpendicular OD; and from (2) that there are two points K and K', equidistant from D, which fulfil the conditions. It also follows that the given line must not cut the circle, as limited in the enunciation; for otherwise the radical part of (2) would be imaginary.

Scholium.—Some years ago I published, in *Leybourn's Repository*, a spherical theorem analogous to this. (*See* vol. vi. p. 133.)

PORISM II.

A straight line may be found, from any point of which tangents being drawn to two given circles, shall be equal to each other.

[Leslie's Geom. Analysis, iii. prop. 25.]

Let the system be referred to the line joining the centres of the given circles, as axis of y, and a line through one of the centres perpendicular to the former, as axis of x. Put the radius of this circle r_1, and that of the other r_2; and let the distance between the centres be k. Also, let the equation of the porismatic line be

$$y = px + q \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and denote the arbitrary point in (1), from which the equal tangents are drawn, by $x'y'$; then

$$y' = px' + q \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

* The expression for the length of the tangent from the given point $x_1\,y_1$ to the circle $(x-h)^2 + (y-k)^2 = \rho^2$, is readily obtained in several ways by geometrical considerations: but strict purity of method requires us to find it without appealing to geometry at all, after our first principles are laid down. The following process is perhaps the most natural and obvious one:—

Find the equation of the tangent through $x_1\,y_1$ (vol. ii. p. 283); then the co-ordinates of the points of contact of the circle and its tangent; and, lastly, the length of the line joining these points (ii. 253).

and we have the squares of the tangents to the given circles drawn from the point $x'y'$, expressed by

$$x'^2 + y'^2 - r^2_1, \text{ and } x'^2 + (y' - k)^2 - r^2_2.$$

Whence equating these and reducing, we have

$$2ky' - k^2 = r^2_1 - r^2_2 \dots\dots\dots\dots (3)$$

Eliminate y' from (2, 3); then there results

$$2kpx' + 2kq = k^2 + r^2_1 - r^2_2.$$

Equating to zero, the coefficient of x', we get the conditional equations

$$2kp = 0, \text{ or } p = 0, \text{ and}$$

$$q = \frac{k^2 + r^2_1 - r^2_2}{2k}.$$

Whence the equation of the porismatic line (1) is expressed by the equation

$$y = \frac{k^2 + r^2_1 - r^2_2}{2k}.$$

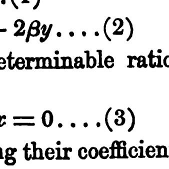
Scholium.—The enunciation given by Leslie is too restricted, as he requires that the circles shall not meet each other. In fact, his *analysis* is only adapted to that case; though when the circles cut each other, the tangents drawn from any point beyond the extremities of their common chord, will be equal to one another. The line in question is the *radical axis* of the two given circles.

PORISM III.

A circle whose centre is O, *and a point* A *being given, another point* P *may be found, such that if lines* AC, PC *be inflected to any point* C *in the circle, they shall have a constant and determinable ratio.*

[Playfair, prop. 2, and Simson de porismatibus, pr. 2.]

Through O, the centre of the circle, draw OA, and perpendicular to OA draw OY, and take these as axes of x and y. Put OA $= a$, the radius equal to r, and denote the porismatic point P by $\alpha\beta$, and the arbitrary point C by xy. Then the equation of the circle is



$$x^2 + y^2 = r^2,$$

$$AC^2 = (x - a)^2 + y^2 = r^2 - 2ax + a^2 \dots (1)$$

$$CP^2 = (\alpha - x)^2 + (\beta - y)^2 = r^2 + \alpha^2 + \beta^2 - 2\alpha x - 2\beta y \dots (2)$$

But by the porism $CP^2 = m^2 AC^2$, where m is the determinable ratio; whence from (1, 2)

$$r^2 + \alpha^2 + \beta^2 - 2(\alpha x + \beta y) - m^2 (r^2 + a^2) + 2m^2 ax = 0 \dots (3)$$

But in this x and y are indeterminate, and hence equating their coefficients to zero, we have

$$r^2 + \alpha^2 + \beta^2 - m^2 r^2 - m^2 a^2 = 0 \dots\dots\dots\dots (4)$$

$$2m^2 a - 2\alpha = 0 \dots\dots\dots\dots (5)$$

$$2\beta = 0 \dots\dots\dots\dots (6)$$

From (6) we have $\beta = 0$, or the point P is in the axis of x, a circumstance *assumed* in all the geometrical solutions with which I am acquainted, and

expressly so by Playfair. It would, probably, be extremely difficult, if not impossible, to treat the porism in so general a form by the strictly geometrical method.

From (5) we have

$$\alpha = m^2 a \ldots\ldots\ldots\ldots(7)$$

Insert (6, 7) in (4); then we get after slight reduction

$$(1 - m^2)(r^2 - m^2a^2) = 0.$$

Whence we have, either

$$r^2 - m^2a^2 = 0, \text{ or } m = \frac{r}{a} \ldots.(8)$$

$$1 - m^2 = 0, \text{ or } m = 1 \ldots.(9)$$

The equation (9) is a real case of the porism, with this exception, that it gives the same point A instead of "another point" P; in which case the ratio is one of equality. The other equation (8) gives

$$\alpha = m^2 a = \frac{r^2}{a^2}\cdot a = \frac{r^2}{a};$$

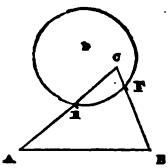
whence all the unknowns are found, and the porism is completely solved.

PORISM IV.

Two straight lines, AC, BC, *and two points* A, B *in them being given, a point* D *may be found, such that if it be made the centre of any circle whatever which cuts the given lines in* E *and* F, *the segments* AE, BF *shall have a determinable constant ratio.*

[Noble, Math. Repos. vol. i. por. 2.

Let the given lines be taken as axes of co-ordinates, and make AC $= a$, BC $= b$; and let $\alpha\beta$ denote the porismatic centre, r the variable radius of the circle, and $m : n$ the determinable ratio. Then the equation of the circle will be



$$(x - \alpha)^2 + 2(x - \alpha)(y - \beta)\cos C + (y - \beta)^2 = r^2 \ldots\ldots(1)$$

The segments CE, CF will be determined by making y and x respectively equal to zero; thus giving

$$\left.\begin{array}{l} \text{AE} = a - \alpha - \beta\cos C \pm \sqrt{r^2 - \beta^2\sin^2 C} \\ \text{BF} = b - \beta - \alpha\cos C \pm \sqrt{r^2 - \alpha^2\sin^2 C} \end{array}\right\} \ldots\ldots\ldots(2)$$

then, by the proposition, we get from (2) the equation

$$n\{a-\alpha-\beta\cos C \pm \sqrt{r^2-\beta^2\sin^2 C}\} = m\{b-\beta-\alpha\cos C \pm \sqrt{r^2-\alpha^2\sin^2 C}\} \ldots(3)$$

Now that this may be universally true for all values of r, we must have the rational parts and irrational parts respectively equal to zero, on the principle of *congruity* (*see* p. 13 of this number); and hence we have

$$na - n\alpha - n\beta\cos C = mb - m\beta - m\alpha\cos C \ldots\ldots(4)$$

$$n\sqrt{r^2 - \beta^2\sin^2 C} = m\sqrt{r^2 - \alpha^2\sin^2 C} \ldots\ldots\ldots\ldots(5)$$

And, again, because r is arbitrary, we have from (5)

$$n - m = 0 \ldots\ldots(6);\ n^2\beta^2 - m^2\alpha^2 = 0 \ldots\ldots(7)$$

Now (6) shews that $n=m$, or the determinable ratio is one of equality; and hence, also, (7) becomes

$$(\alpha-\beta)(\alpha+\beta)=0 \qquad (8)$$

Also, putting $m=n$ in (4) we find

$$\alpha-\beta=\frac{a-b}{1-\cos C} \qquad (9)$$

Combine (9) with $\alpha+\beta=0$, deduced from (8), and we shall obtain

$$\alpha=\frac{a-b}{4\sin^2\frac{1}{2}C}, \text{ and } \beta=-\frac{a-b}{4\sin^2\frac{1}{2}C}.$$

If now we take the other factor of (8), that is $\alpha-\beta=0$, and combine it with (9), we get an apparent absurdity: namely, α and β indeterminate. At least, it would seem not to be a part of the solution. As, however, we have employed no elimination, it cannot be a foreign factor; and hence we may infer that it belongs to some peculiar case of the conditions of the porism.

The following considerations will shew that this is an essential part of the solution. If we compare together $\alpha-\beta=0$ and $\alpha-\beta=\frac{a-b}{1-\cos C}$, we shall see that both are fulfilled by $a-b=0$, giving the triangle isosceles. And, again, α, β will be indeterminate when $1-\cos C=0$, or the two lines coincide, and the two points A, B likewise coincide; for then $\alpha-\beta=\frac{0}{0}$, and hence α and β are themselves separately expressible by $\frac{0}{0}$.

As these cases are possible ones, the solution would have been incomplete without a provision for their existence; and it is the peculiar province of *algebraic investigations* to give solutions adapted to every possible case, by means of the same general equations.

Scholium.—The principle of the radical being itself an *indeterminate*, will often, as in the case above, simplify the reductions considerably; as the reader may convince himself by eliminating them from (3) before he equates the coefficients to zero, when he will perceive the great amount of labour saved by the process here employed.

PORISM V.

Let A,B *be given points, and* CDE *a given circle: a point* P *can be found, such that if from* P *any line be drawn to cut the circle in* C,D; *then the four points* A,B,C,D *shall be in one circumference.*

[Noble, Math. Comp. vol. i. por. 4.]

Join AB, and bisect it in O; and draw OY perpendicular to it; and take OA, OY as axes of x and y. Put $OA=OB=a$; also let h k be the centre and r the radius of the given circle CDE. Then its equation is

$$y^2+x^2-2ky-2hx=r^2-h^2-k^2 \qquad (1)$$

Then also *any circle* through A, B, will be

$$y^2+x^2+y'y=a^2 \qquad (2)$$

where y' is arbitrary.

Now the circles (1, 2) intersect in the line which, by hypothesis, passes

through $\alpha\beta$, the porismatic point P. Whence, subtracting (2) from (1), we have, putting $\alpha\beta$ for xy, in the result,

$$\beta y'+2(k\beta+h\alpha)=a^2+h^2+k^2-r^2 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Equating to zero, the coefficients of the indeterminate y', we get from (3)

$$\beta=0, \text{ and } 2(k\beta+h\alpha)=a^2+h^2+k^2-r^2 \ldots\ldots\ldots\ldots (4, 5)$$

The equation (4) shews that the point P is in the line AB; and (4) inserted in (5) gives

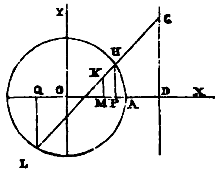
$$\alpha=\frac{a^2+h^2+k^2-r^2}{2h}.$$

Whence, the porism is true, and the point P is determined.

PORISM VI.

A circle AHL *and a straight line* DG *being given; a point* K *may be found, such that if any chord whatever* HL *be drawn through* K, *meeting the circle in* H,L, *and the line* DG *in* G, *the following relation will subsist:*

$$KH.LG = LK.HG.$$



Through O the centre of the circle draw the perpendicular, OX, to GD meeting it in D, and OY parallel to DG; and take these as axes of x and y. Let the radius be denoted by r, OD by a; and the porismatic point K by $\alpha\beta$. Then the equation of HL, p being arbitrary, will be

$$y-\beta=p(x-\alpha) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also, the equation of the circle is

$$x^2+y^2=r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The intersections of (1, 2) give for the values of x

$$x=\frac{p(\alpha p-\beta)\pm R}{1+p^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\text{where, } R^2=(1+p^2)\,r^2-(\alpha p-\beta)^2 \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The upper sign obviously refers to the point H most remote from the origin, in the positive direction, and the lower to the other point L; so that

$$OP=\frac{p(\alpha p-\beta)+R}{1+p^2}, \text{ and } OQ=\frac{p(\alpha p-\beta)-R}{1+p^2}.$$

Again, when HL cuts the line DG, we have $x=a$, and hence the segments of the axis of x corresponding to the segments of HL, specified in the enunciation, are, when HL cuts the circle on the same side of the axis of y,

$$QM=\alpha-\frac{p(\alpha p-\beta)-R}{1+p^2}=\frac{R+(\alpha+p\beta)}{1+p^2} \ldots\ldots\ldots (5)$$

$$QD=a-\frac{p(\alpha p-\beta)-R}{1+p^2}=\frac{a+(a-\alpha)p^2+p\beta+R}{1+p^2} \ldots (6)$$

$$PM=\frac{p(\alpha p-\beta)+R}{1+p^2}-\alpha=\frac{R-(\alpha+p\beta)}{1+p^2} \ldots\ldots\ldots (7)$$

$$PD=a-\frac{p(\alpha p-\beta)+R}{1+p^2}=\frac{a+(a-\alpha)p^2+p\beta-R}{1+p^2} \ldots (8)$$

But since KH.LG = LK.HG, we have also, QD.MP = QM.PD *; inserting in this from (5, 6, 7, 8), we have the two sides of the equation respectively represented by

$$\{a+(a-\alpha)p^2+p\beta\}R-\{a+(a-\alpha)p^2+p\beta\}(\alpha+p\beta)+R^2-R(\alpha+p\beta),$$

$$\{a+(a-\alpha)p^2+p\beta\}R+\{a+(a-\alpha)p^2+p\beta\}(\alpha+p\beta)-R^2-R(\alpha+p\beta);$$

$$\text{whence, } R^2=\{a+(a-\alpha)p^2+p\beta\}(\alpha+p\beta)\ldots\ldots\ldots\ldots\ldots(9)$$

Restoring the value of R^2 from (4), and arranging according to the arbitrary quantity p, we get

$$(a-\alpha)\beta p^3+(a\alpha+\beta^2-r^2)p^2+(a-\alpha)\beta p+(a\alpha+\beta^2-r^2)=0\ldots(10)$$

Whence equating coefficients, as in the former porisms,

$(a-\alpha)\beta=0\ldots\ldots\ldots(11)$	$\alpha a+\beta^2-r^2=0\ldots\ldots\ldots(12)$
$(a-\alpha)\beta=0\ldots\ldots\ldots(13)$	$\alpha a+\beta^2-r^2=0\ldots\ldots\ldots(14)$

which are obviously but one pair of equations. Taking, then, (11, 12), we have

$$\beta=0, \text{ and } \alpha=\frac{r^2}{a}.$$

Scholia.

1. Had it been *assumed*, as is done in all the geometrical solutions of this porism, that the point K is in the line OD, the process would have been very much simplified; and it may be added that if this assumption be not made the geometrical solution will become one of extreme difficulty.

2. The point K thus determined is the *pole* corresponding to the *polar* DG, in respect of the given circle.

3. Exactly the same process, with scarcely more complexity, will establish the property for the ellipse and hyperbola, which referred to conjugate diameters, one of which is parallel to DG, are represented by

$$a^2y^2 \pm b^2x^2 = a^2b^2$$

* The employment of this principle might be supposed, at first sight, to be a return to the *doctrine* of *similar triangles* for our intermediate steps of investigation; and as it is one which not only occurs in this, but in other classes of inquiry (especially i those which relate to segments of the same line) into which the co-ordinate system enters, it is desirable to deduce it from the ordinary equation of the straight line. The following method fully answers the purpose.

Let x_1y_1, x_2y_2, x_3y_3, denote any three points A, B C, in the straight line

$$y=ax+b\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

Then, if the angle of ordination be A, we have the usual expressions

$$AB^2=(x_2-x_1)^2+2(x_2-x_1)(y_2-y_1)\cos A+(y_2-y_1)^2\ldots\ldots\ldots(b)$$

$$BC^2=(x_3-x_2)^2+2(x_3-x_2)(y_3-y_2)\cos A+(y_3-y_2)^2\ldots\ldots\ldots(c)$$

But since these points are in (1) we have

$$y_1=ax_1+b;\ y_2=ax_2+b;\ \text{and } y_3=ax_3+b:\ \text{whence}$$

$$y_2-y_1=a(x_2-x_1),\ \text{and } y_3-y_2=a(x_3-x_2).$$

Insert these values in (b, c): then

$$AB=(x_2-x_1)\{1+2a\cos A+a^2\}^{\frac{1}{2}}\ldots\ldots\ldots\ldots(d)$$

$$BC=(x_3-x_2)\{1+2a\cos A+a^2\}^{\frac{1}{2}};\ldots\ldots\ldots\ldots(e)$$

wherefore, we obtain the relation in question, by division of (d) by (e); viz.

$$\frac{AB}{BC}=\frac{x_2-x_1}{x_3-x_2}.$$

The same principle is extensible to any number of points; and in a similar manner to the co-ordinates *of the axis of y. It is* also applicable, and proved in a corresponding *manner, when the system of points* is referred to three axes in space.

$$\frac{x'}{x'-h}=\frac{m}{n};\text{ or again, }\frac{m}{m-n}=\frac{x'}{x'-(x'-h)}=\frac{x'}{h}=\frac{c(k-b)}{ck-hb}\,..(5)$$

But by the porism, the circle (1) passes through $\alpha\beta$, and hence

$$\alpha^2+2\alpha\beta\cos A+\beta^2-h\alpha-k\beta=0\,\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Eliminate h from (5, 6), and then, since k is indeterminate, we have

$$mb\,(\alpha^2+2\,\alpha\beta\cos A+\beta^2)-(m-n)\,bc\alpha=0\,\ldots\ldots\ldots\ldots(7)$$

$$nc\alpha+mb\beta=0\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Whence, from (7, 8) we get

$$\alpha=\frac{(m-n)\,mb^2c}{m^2b^2-2mnbc\cos A+n^2c^2}\,\ldots\ldots\ldots\ldots(9)$$

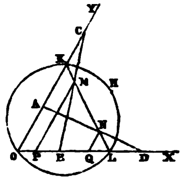
$$\beta=\frac{(n-m)\,nbc^2}{m^2b^2-2mnbc\cos A+n^2c^2}\,\ldots\ldots\ldots\ldots(10)$$

Wherefore the point H is determined; and it is expressed, as we might expect, by symmetrical functions of m and n, and of b and c.

PORISM X.

Let OC, OD, BC, AD *be four lines given by position: a point* H *may be found, such that if through it and any one of the six intersections of the lines, as* O, *a circle be described to meet the lines through whose intersection it passes in* K *and* L; *then* KL *will be divided by the remaining lines in* M *and* N, *so that the segments shall have constant ratios, and which ratios can be found.*

[Wallace, prop. vi.]



Let $OA=a$, $OB=b$, $OC=c$, $OD=d$; and $XOY=\theta$, then proceeding as in the last porism, we shall have the equations of KL, BC, AD; viz.

$$hy+kx=hk\,\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$by+cx=bc\,\ldots\ldots\ldots\ldots\ldots\ldots(2)$$
$$dy+ax=ad\,\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The intersection of (1, 2) and that of (1, 3) give for the abscisses of M and N respectively,

$$x_1=\frac{hb(k-c)}{bk-ch},\text{ and }x_2=\frac{hd(k-a)}{dk-ah}\,\ldots\ldots\ldots\ldots(4)$$

If now the given ratios be represented by m and n, and we adopt the principle of the note at page 51, we shall have

$$m=\frac{LQ}{LO}=\frac{LO-OQ}{LO}=\frac{a(d-h)}{dk-ah},$$

$$n=\frac{LP}{LO}=\frac{LO-OP}{LO}=\frac{c(b-h)}{bk-ch}.$$

From these reduced we get the equations

$$mdk=(m-1)\,ah+ad\,\ldots\ldots\ldots\ldots\ldots\ldots(5)$$
$$nbk=(n-1)\,ch+bc\,\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Let, moreover, $\alpha\beta$ denote the porismatic point H: then

$$\alpha^2+2\alpha\beta\cos\theta+\beta^2-h\alpha-k\beta=0\,\ldots\ldots\ldots\ldots(7)$$

Substitute the values of k from (5) and (6) separately in (7); then there result the two equations

$$\{md\alpha+(m-1)\,a\beta\}\,h-\{md\,(\alpha^2+2\alpha\beta\cos\theta+\beta^2)-ad\beta\}=0,$$
$$\{nb\alpha+(n-1)\,c\beta\}\,h-\{nb\,(\alpha^2+2\alpha\beta\cos\theta+\beta^2)-bc\beta\}=0$$

In both of which h is indeterminate, and hence its coefficients being separately equated to zero, we obtain

$$m(d\alpha+a\beta)=a\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$n(b\alpha+c\beta)=c\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

$$md(\alpha^2+2\alpha\beta\cos\theta+\beta^2)=ad\beta \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$nb(\alpha^2+2\alpha\beta\cos\theta+\beta^2)=bc\beta \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

From (10, 11) we get, by division, $na=mc \ldots\ldots\ldots(12)$

Whence, multiplying (8) by c and (9) by a, we get,

$$mc(d\alpha+a\beta)=ac\beta, \text{ and } na(b\alpha+c\beta)=ac\beta,$$

whence $d\alpha+a\beta=b\alpha+c\beta$; or $(d-b)\alpha+(a-c)\,\beta=0 \ldots(13)$

Insert the value of β from (13) in (8), and we find by (12), that

$$d\alpha+a\beta=\frac{ab-cd}{a-c}\,\alpha,$$

$$\text{and } m=\frac{a\beta}{d\alpha+a\beta}=\frac{(b-d)a}{ab-cd} \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

Similarly, from (13) and (9) is obtained

$$n=\frac{c\beta}{b\alpha+c\beta}=\frac{(b-d)c}{ab-cd} \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

Insert in (10) the value of m from (14), and the value of α in terms of β from (13): then, after slight reduction,

$$\beta=\frac{(b-d)(ab-cd)}{(a-c)^2+2\,(a-c)(b-d)\cos\theta+(b-d)^2} \ldots(16)$$

$$\text{and hence, } \alpha=\frac{(a-c)(ab-cd)}{(a-c)^2+2\,(a-c)(b-d)\cos\theta+(b-d)^2} \ldots(17)$$

Scholium.—The point H of this proposition has a considerable number of remarkable properties. A few of them are enunciated in *Hutton's Course*, vol. ii. page 244:—the point P in that place answering to H in the investigation above. See also, *Leybourn's Repository*, vol. i. p. 170, and vol. vi. pp. 126 and 234; and *Quetelet's Correspondance Mathématique et Physique*, tom. v. p. 218.

PORISM XI.

Given three right lines G, A, B; *there is given a fourth line* R *such that if* r, q, p, *be the base, perpendicular and hypothenuse of a right angled triangle, and* X *be a fourth proportional to the square on* q, *the square on* p *and the right line* R; *and* Z *a fourth proportional to the right lines* q, r, *and* B; *then if* P *be a mean proportional between* G *and the sum of the right lines* A, X, Z, *and* Q *a mean proportional between* G *and the excess of* A *and* X *above* Z, *the sum of* P *and* Q *shall be a given magnitude* S. [Gompertz, Imaginary Quantities, p. 32.]

The conditions of this porism, symbolically expressed, are:

$$X=\frac{p^2R}{q^2} \ldots\ldots(1) \qquad P=\sqrt{G\,(A+X+Z)} \ldots\ldots(4)$$

$$Z=\frac{rB}{q} \ldots\ldots(2) \qquad Q=\sqrt{G\,(A+X-Z)} \ldots\ldots(5)$$

$$p^2=q^2+r^2 \ldots\ldots(3) \qquad P+Q=S \ldots\ldots(6)$$

where p and q are indeterminate, and R and S are to be found.

Substitute (1, 2) in (4, 5), and in these terms express (6): this gives

$$\frac{S}{\sqrt{G}} = \left\{ A + \frac{p^2R}{q^2} + \frac{rB}{q} \right\}^{\frac{1}{2}} + \left\{ A + \frac{p^2R}{q^2} - \frac{rB}{q} \right\}^{\frac{1}{2}}.$$

Square and transpose: then

$$\frac{S^2}{G} - 2\left(A + \frac{p^2R}{q^2}\right) = 2\left\{\left(A + \frac{p^2R}{q^2}\right) - \frac{r^2B^2}{q^2}\right\}^{\frac{1}{2}};$$

and squaring, cancelling, inserting (5), and arranging in p and q, we get

$$q^2(S^4 - 4AGS^2 - 4G^2B^2) + 4p^2(B^2G^2 - S^2RG) = 0 \ldots .(7)$$

But since p and q are indeterminate, this becomes

$$S^4 - 4AGS^2 - 4G^2B^2 = 0 \ldots\ldots\ldots\ldots(8)$$

$$B^2G^2 - S^2RG = 0 \ldots\ldots\ldots\ldots(9)$$

From (8) we have

$$S^2 = 2\{A \pm \sqrt{A^2 + B^2}\}G \ldots\ldots\ldots\ldots(10)$$

the lower sign of which is inadmissible, since the use of it would render S imaginary.

From (9, 10) we get

$$R = \frac{B^2G}{S^2} = \frac{B^2}{2\{A + \sqrt{A^2 + B^2}\}} \ldots .(11)$$

Scholium.—This porism is very simple, according to this method of solution. It is, notwithstanding, interesting for the geometrical solution given by Mr. Gompertz, in the work referred to. I cannot, however, view his "case of case" as being, strictly speaking, *imaginary*, except we trammel ourselves with the unnecessary use of the term right-angled-triangle: the condition being only in strictness, that $p^2 = r^2 + q^2$, p, q, and r being any numbers (zero included) which fulfil the equation.

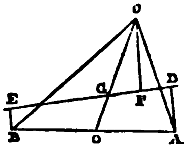
The view entertained by Mr. Gompertz relative to porisms, though only incidentally introduced into this tract, is extremely ingenious. It consists in this:—

That whenever an algebraical expression is given, one or more arbitrary quantities may be introduced into the expression without altering its value. The process of finding the forms under which these arbitrary quantities appear, or of assigning the conditional equations, he calls *porismatising the expression.* Mr. Gompertz also employs the same principle in justifying the use of the imaginary symbol $\sqrt{-1}$, by finding how the real part of the expression is to be formed so as to substitute $\sqrt{\rho - 1}$ for $\sqrt{-1}$, the value of the final expression retaining the value it originally had. As, however, we shall recur to this subject, we shall not enter into further detail on the present occasion: inasmuch as the originality and importance of the principle he has employed, deserves more expanded consideration than could be devoted to it in a scholium.

Thus, for instance, if there be given a and b, a value may be found for k, such that if $m = \frac{k\rho}{\sqrt{a^2 - \rho^2}}$, then $\sqrt{(a^2 - \rho^2)(b^2 + m^2)} = ab$, whatever the value of ρ may be. Or, again, if $k = \frac{m}{b + \rho}$, and $m = \frac{k\rho}{b + \rho}$, a value may be found for k which will render $(a - m)(b + \rho) = ab$, for all values of ρ.

PORISM XII.

The triangle ABC *being given: a point* G *may be found, such that drawing through it any straight line whatever, and from* A, B, C, *the perpendiculars (or any three parallel lines)* AD, CF, BE *to meet the line in* D, F, E; *then the sum of the two perpendiculars falling upon one side of the arbitrary line will be equal to the perpendicular falling on the other side of it.* [Playfair, Art. 10.]



Bisect AB in O; and take OA, OC as axes of x and y; then the points A, B, C will be expressed by $(a, 0)$, $(-a, 0)$, and $(0, c)$. Also denote the porismatic point G by $\alpha\beta$, and the arbitrary line DE by

$$y-\beta=p(x-\alpha) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

the arbitrary quantity being p.

Then, (*Hutton*, vol. ii. p, 263), we have (putting $\omega=$ COA)

$$\text{AD}=\frac{\pm\{-pa-(\beta-p\alpha)\}\sin\omega}{\sqrt{1+2p\cos\omega+p^2}}\ldots(2)$$

$$\text{BE}=\frac{\pm\{+pa-(\beta-p\alpha)\}\sin\omega}{\sqrt{1+2p\cos\omega+p^2}}\ldots(3)$$

$$\text{CF}=\frac{\mp\{c-(\beta-p\alpha)\}\sin\omega}{\sqrt{1+2p\cos\omega+p^2}}\ldots(4)$$

the sign of the last being the contrary of the other two which fall on the opposite side of the line.*

* Much discussion has arisen relative to the signification of the double sign in the algebraic expression for the length of a line. I have more than once expressed my own view on this question, to this effect:—that as the absolute length may be measured from either extremity of the line in the direction of the other, and as there does not arise from the hypothesis introduced any reason whatever for preferring one extremity rather than the other, as the origin; there ought to be the same degree of ambiguity in the expression that there was in the original hypothesis, and hence, the double sign is essential to the *general* expression of that length. The one which we use as adapted to any specified case, merely for the sake of convenience, like the sign of direction in the co-ordinate axes, ought, however, when once adopted, to be adhered to throughout the entire investigation. The example above illustrates this, and was inserted here more for the sake of this note, than for its intrinsic value or difficulty,—the solution given in the following porism being in all respects else, superior to this in elegance and general character.

In co-ordinates we mean by $+$ and $-$ to designate the *direction* in which we pass from one point to another, the mode of passage being always of the *same kind*. Now, if from two points on the same side of a line perpendiculars be drawn, the same mode of estimation will require that we either consider them both drawn from the points to the line, or both from the line to the points; and in either case, the passages are of the same kind as to direction: but if in one case we suppose the perpendicular drawn from the point to the line, and the other drawn from the line to the point, our fundamental notions lead to one statement of the directions being opposite; and if we should have occasion to express this algebraically, we should mark these two perpendiculars with opposite signs, either $\pm$, $\mp$, or $\mp$, $\pm$. It will follow, then, that if from two points on different sides of the line we draw perpendiculars, and we wish to express these *consistently* with one general principle, we must mark these perpendiculars with *opposite signs*. In fact, for the expression of a single perpendicular, though it *might not* affect the final result to use only the sign $+$, we ought for generality to retain the double sign: but when we have to consider several which may originate in *any* point, it becomes imperative that we should retain it. In all cases, too, where a line so determined is to be taken in connection with another line, the double sign is imperative, as the line thus expressed may either increase or diminish another given line. See, for instance, the determination of K and K′, in the first porism; and numerous similar instances will present themselves to the reader's mind at once.

Forming now the equation of the porism, and reducing, we get

$$3pa - 3\beta + c = 0 \ldots\ldots\ldots (5)$$

Whence, since p is indeterminate, we have

$$a = 0, \text{ and } \beta = \frac{1}{3} c; \ldots (6, 7)$$

or the point G is in the line OC, and OG is one-third of OC.

Scholium—The point now found, as is well known, is that usually, but improperly, *in geometry* at least, named the centre of gravity of the triangle ABC. Carnot has proposed as a geometrical name, *le centre des moyennes distances*: but whilst this is inconveniently lengthy as a term, it is also objectionable from its implication of an erroneous theorem, when taken generally. The term *centroïd* seems to answer all purposes, and to be free from every possible objection. This term has, indeed, been used to indicate another point in reference to the circle, by Dr. Hey (*Phil. Trans.* 1814), but, as far as I know, by him only; and as that point is now generally known, both in this country and on the continent, by the term *pole*, it is useless to retain the word centroïd for that purpose, when it can be used so much more advantageously in a case of constant occurrence, and where an adequate term is so much required.

The proposition in this porism will be treated in a different way in the following solution of the general property.

PORISM XIII.

Any number of points $A_1, A_2, \ldots A_n$, *and as many magnitudes* $a_1, a_2, \ldots a_n$ *being given: a point may be found, such that upon any straight line drawn through it, perpendiculars* $p_1, p_2, \ldots p_n$ *be drawn, then (having regard to the signs of these, as in the preceding note), we shall have*

$$a_1 p_1 + a_2 p_2 + \ldots + a_n p_n = 0.$$

Denote the given points by $x_1 y_1$, $x_2 y_2 \ldots x_n y_n$, and the porismatic point by $a\beta$; then the arbitrary line will be *

$$y - \beta = (x - a) \tan \omega,$$

where ω is arbitrary; and the perpendiculars will be

$$p_1 = (y_1 - \beta) \cos \omega - (x_1 - a) \sin \omega,$$
$$p_2 = (y_2 - \beta) \cos \omega - (x_2 - a) \sin \omega,$$
$$\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots$$
$$p_n = (y_n - \beta) \cos \omega - (x_n - a) \sin \omega.$$

Forming now the equation of the porism, and equating [illegible] efficients of the arbitraries sin ω and cos ω, we have

$$a_1(y_1 - \beta) + a_2(y_2 - \beta) + \ldots\ldots\ldots + a_n(y_n -$$
$$a_1(x_1 - a) + a_2(x_2 - a) + \ldots\ldots\ldots + a_n(x$$

[illegible] in which to take the equation of the straight line, [illegible] it are to be considered, is $y = x \tan \omega + b$, as the [illegible] is most simply expressed, by $p = \mp (y_1 - b) \cos \omega -$ [illegible]ly omitted in *writing* the investigation of this porism, since [illegible]

and hence the co-ordinates of the point which we have defined as the *centroïd*, are found to be

$$\beta=\frac{a_1y_1+a_2y_2+\ldots.+a_ny_n}{a_1+a_2+\ldots.+a_n},$$

$$\alpha=\frac{a_1x_1+a_2x_2+\ldots.+a_nx_n}{a_1+a_2+\ldots.+a_n}.$$

PORISM XIV.

Let there be any number of lines given in position, and from any point in one of them, perpendiculars be drawn to all the rest: a point may be found, such that the square of the line joining it and the arbitrary point from which the perpendiculars are drawn, shall have to the sum of the squares of those perpendiculars a constant ratio, which can be found. [Playfair, Art. 26.]

Let there be $(n+1)$ lines given; and take the origin of rectangular co-ordinates at any point in that from which the perpendiculars are drawn, and that line itself as the axis of x. Then the equation of the line itself is

$$y=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Also let the other n given lines be expressed by

$$\left.\begin{array}{l} y=x\tan\omega_1+b_1,\\ y=x\tan\omega_2+b_2,\\ \ldots\ldots\ \ldots\ldots\ldots\\ y=x\tan\omega_n+b_n.\end{array}\right\}\ldots\ldots(2)$$

Thus the lines from the arbitrary point $x0$ in (1) perpendicular to the given lines (2) and to the porismatic point $\alpha\beta$ being expressed, then (m being the porismatic ratio) the equation of the porism will be

$$(x\sin\omega_1+b_1\cos\omega_1)^2+(x\sin\omega_2+b_2\cos\omega_2)^2+\ldots=m^2\{(\alpha-x)^2+\beta^2\}\ \ldots(3)$$

Collect and equate to zero, the coefficients of the homologous terms of the arbitrary quantity x; then there result the equations

$$\sin^2\omega_1+\sin^2\omega_2+\sin^2\omega_3+\ldots+\sin^2\omega_n=m^2\ldots\ldots\ldots\ldots\ldots(4)$$

$$2b_1\sin\omega_1\cos\omega_1+2b_2\sin\omega_2\cos\omega_2+\ldots+2b_n\sin\omega_n\cos\omega_n=-2m^2\alpha\ldots(5)$$

$$b_1^2\cos^2\omega_1+b_2^2\cos^2\omega_2+b_3^2\cos^2\omega_3+\ldots+b_n^2\cos^2\omega_n=m^2(\alpha^2+\beta^2)\ldots(6)$$

The equation (4) gives the porismatic ratio, m; and from (5) we get

$$\alpha=-\frac{b_1\sin 2\omega_1+b_2\sin 2\omega_2+\ldots.+b_n\sin 2\,\omega_n}{2\{\sin^2\omega_1+\sin^2\omega_2+\ldots.+\sin^2\omega_n\}}\ldots\ldots\ \ldots.(7)$$

And from (6) we have by means of (4) and (5),

$$\beta^2=\frac{b_1^2\cos^2\omega_1+\ldots.+b_n^2\cos^2\omega_n}{\sin^2\omega_1+\ldots.+\sin^2\omega_n}-\left\{\frac{b_1\sin 2\omega_1+\ldots+b_n\sin 2\omega_n}{2(\sin^2\omega_1+\ldots+\sin^2\omega_n)}\right\}^2.$$

Whence the porismatic point is found. It is, however, to shew the truth of the porism, necessary to prove that this value of β^2 is always positive, or β always real.

For $\sin 2\omega_1$, write its value $2\sin\omega_1\cos\omega_1$, and the like with all the others:

From (1, 3) we have for finding the distance NM, the elements

$$x-a=\frac{(y_2-\beta)x'-(ay_2-\beta x_2)}{px'+(y_2-px_2)}\dots\dots(4)$$

$$y-\beta=\frac{(y_2-\beta)x'-(ay_2-\beta x_2)}{px'+(y_2-px_2)}.\,p\dots\dots(5)$$

The distance MN, denoting the angle of ordination by C and dropping the accent from the indeterminate x', is

$$\text{MN}=\frac{(y_2-\beta)x-(ay_2-\beta x_2)}{px+(y_2-px_2)}\left\{1+2p\cos\text{C}+p^2\right\}^{\frac{1}{2}}\dots\dots(6)$$

Again, when AE meets OK, we have $x=0$ in (2), and hence we get, dropping the accent as before,

$$\text{GH}=\frac{(a+y_1)x-ax_1}{x-x_1}\dots\dots(7)$$

Put $\text{R}^2=1+2p\cos\text{C}+p^2\dots\dots(8)$

then the condition of the porism is, that

$$\frac{(y_2-\beta)x-(ay_2-\beta x_2)}{px+(y_2-px_2)}.\,\text{R}:\frac{(a+y_1)x-ax_1}{x-x_1}::m:n\dots(9)$$

or reducing and arranging in terms of x, it becomes, the coefficients being equated to zero,

$$n\text{R}(y_2-\beta)-m(a+y_1)p=0\dots\dots(10)$$

$$n\text{R}(y_2-\beta)x_1+n\text{R}(ay_2-\beta x_2)+m(a+y_1)(y_2-px_2)-mapx_1=0\dots(11)$$

$$n\text{R}(ay_2-\beta x_2)x_1+max_1(y_2-px_2)=0\dots\dots(12)$$

From (10) and (12) we get

$$n\text{R}(y_2-\beta)=m(a+y_1)p\dots\dots(13)$$

$$\text{and } n\text{R}(ay_2-\beta x_2)=-ma(y_2-px_2)\dots\dots(14)$$

which inserted in (11) gives, after cancelling common terms,

$$p=\frac{y_2}{x_2-x_1}\dots\dots(15)$$

From this very simple result, we see that the inclination of the porismatic line is dependent solely on the positions of the given points A, B, whatever the ratio $m:n$ and the position of H may be.

Also from (8, 15) the value of R is found, and R is, therefore, now a known quantity.

Again, from (13) and (15) we find at once

$$\beta=y_2-\frac{m(a+y_1)p}{n\text{R}}=\frac{y_2}{x_2-x_1}.\frac{n\text{R}(x_2-x_1)-m(a+y_1)}{n\text{R}}\dots\dots(16)$$

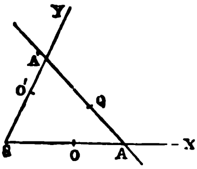
and from (14, 15, 16) we have

$$a=\frac{\beta x_2}{y_2}-\frac{ma(y_2-px_2)}{n\text{R}y_2}=\frac{(x_2-x_1)(n\text{R}x_2-ma)-mx_2y_1}{n\text{R}(x_2-x_1)}\dots(17)$$

The point M is therefore found, and the direction of the line KL through it having been found in (15), the porism is resolved.

PORISM XVII.

Let SO, SO$'$, *be two given lines, and* O, O$'$ *given points in them and* Q *another given point: then values for* λ, μ *may be found, such that if any line* A A$'$ *whatever be drawn through* Q, *we shall have*



$$\lambda . \frac{OA}{SA} + \mu . \frac{O'A'}{SA'} = 1.$$

[Chasles, Aperçu Historique, p. 279.]

Take the given lines as axes of x and y, denote Q by $x_1 y_1$, and let SO$=a$, SO$'=b$: then p being indeterminate, the equation of the line AA$'$ will be

$$y - y_1 = p(x - x_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Hence, $SA = -\dfrac{y_1 - px_1}{p}$, and $SA' = y_1 - px_1 \ldots\ldots\ldots\ldots (2, 3)$

Now the equation of the porism may, obviously, be transformed to

$$\lambda . \frac{SO}{SA} + \mu . \frac{SO'}{SA'} = \lambda + \mu - 1 \ldots\ldots\ldots\ldots (4)$$

Insert the given values of SO, SO$'$, and those of SA, SA$'$ from (2, 3), and equating the coefficients of p to zero: then there result

$$(\lambda + \mu - 1)x_1 = \lambda a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$(\lambda + \mu - 1)y_1 = \mu b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Divide (5) by (6): then

$$\frac{x_1}{y_1} = \frac{\lambda}{\mu} . \frac{a}{b}, \text{ or } \mu = \frac{a\,y_1}{b\,x_1} . \lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Substitute this in (5) or (6), and resolve: then

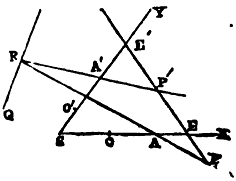
$$\lambda = \frac{bx_1}{bx_1 + ay_1 - ab}, \text{ and } \mu = \frac{ay_1}{bx_1 + ay_1 - ab} \ldots\ldots\ldots\ldots (8, 9)$$

wherefore the position of the line AA$'$ through Q is indeterminate, and the equation fulfilled. The result is

$$bx_1 . \frac{OA}{SA} + ay_1 . \frac{O'A'}{SA'} = bx_1 + ay_1 - ab \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

PORISM XVIII.

Let SO, SO$'$ *be given lines, and* O,O$'$ *given points in them, and let* P, P$'$ *be two other given points, and* RQ *a given line: from any point* R *in* RQ *draw lines* RP, RP$'$, *to meet* SO, SO$'$ *in* A, A$''$, *and join* PP$'$ *meeting* SO, SO$'$ *in* E *and* E$'$: *then values can be found for* λ, μ, *such that wherever in* RQ *the point* R *be taken, the following equation will be fulfilled;*



$$\lambda . \frac{OA}{EA} + \mu . \frac{O'A''}{E'A'} = 1.$$

[Chasles, Aperçu Historique, p. 278.]

Take SO, SO$'$ as axes of x and y respectively; denote the variable point in QR by $x'y'$, and the given points P, P$'$ by $a_1\beta_1$, $a_2\beta_2$; also put SO $= h$, SO$' = k$, and denote the line RQ by

$$y'' = px'' + q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Then the lines PR, P′R, and PP′ will be respectively expressed by

$$y(x'-a_1)-x(y'-\beta_1)=x'\beta_1-y'a_1 \ldots\ldots\ldots(2)$$

$$y(x'-a_2)-x(y'-\beta_2)=x'\beta_2-y'a_2 \ldots\ldots\ldots(3)$$

$$y(a_1-a_2)-x(\beta_1-\beta_2)=a_1\beta_2-a_2\beta_1 \ldots\ldots\ldots(4)$$

When (2) cuts the axis of x, and (3) cuts that of y, we have

$$\text{SA}=\frac{y'a_1-x'\beta_1}{y'-\beta_1} \ldots\ldots\ldots(5)$$

$$\text{SA}'=\frac{x'\beta_2-y'a_2}{x'-a_2} \ldots\ldots\ldots(6)$$

Also, when (4) cuts both the axes of x and y, we have respectively

$$\text{SE}=\frac{\beta_1 a_2-\beta_2 a_1}{\beta_1-\beta_2} \ldots\ldots\ldots(7)$$

$$\text{SE}''=\frac{a_1\beta_2-a_2\beta_1}{a_1-a_2} \ldots\ldots\ldots(9)$$

Whence, from (5) and (6) respectively we obtain

$$\text{OA}=\text{SA}-\text{SO}=\frac{y'a_1-x'\beta_1}{y'-\beta_1}-h=\frac{y'(a_1-h)-\beta_1(x'-h)}{y'-\beta_1} \ldots(9)$$

$$\text{O}'\text{A}'=\text{SA}'-\text{SO}'=\frac{x'\beta_2-y'a_2}{x''-a_2}-k=\frac{x'(\beta_2-k)-a_2(y'-k)}{x'-a_2} \ldots(10)$$

Again, from (5, 7) and from (6, 8) we obtain

$$\text{EA}=\text{SA}-\text{SE}=\frac{y'a_1-x'\beta_1}{y'-\beta_1}-\frac{\beta_1 a_2-\beta_2 a_1}{\beta_1-\beta_2}$$

$$=\frac{\beta_1}{\beta_1-\beta_2}\cdot\frac{y'(a_1-a_2)-x'(\beta_1-\beta_2)+(\beta_1 a_2-\beta_2 a_1)}{y'-\beta_1} \ldots(11)$$

$$\text{E}'\text{A}'=\text{SA}'-\text{SE}'=\frac{x'\beta_2-y'a_2}{x'-a_2}-\frac{a_1\beta_2-a_2\beta_1}{a_1-a_2}$$

$$=\frac{a_2}{a_1-a_2}\cdot\frac{y'(a_2-a_1)-x'(\beta_2-\beta_1)+(a_1\beta_2-\beta_1 a_2)}{x'-a_2} \ldots(12)$$

Hence, by the insertion of (9, 10, 11, 12) in the equation of the porism, and by arranging the result in terms of x' and y', we get

$$a_2 y'\{\lambda(\beta_1-\beta_2)(a_1-h)+\mu\beta_1(a_1-a_2)-\beta_1(a_1-a_2)\}$$
$$-\beta_1 x'\{\lambda a_2(\beta_1-\beta_2)+\mu(a_1-a_2)(\beta_2-k)+a_2(\beta_1-\beta_2)\}$$
$$+a_2\beta_1\{\lambda h(\beta_1-\beta_2)-\mu k(a_1-a_2)-(\beta_1 a_2-\beta_2 a_1)\}=0 \ldots\ldots\ldots(13)$$

Substitute in (13) for y' its value from (1): then there results an equation of the first degree in respect of the indeterminate x', and hence two conditional equations for finding λ and μ.

The result is easily obtained, but the length of the expression precludes its insertion on a page of this breadth.

Scholium.—M. Chasles does not give any investigation of these two porisms, but merely enunciates them, and states that their converses are also true. The converses which he enunciates, are, however, only a common theorem, and a local theorem. Noble, indeed, affirms that the converses of porisms are porisms, which seems also to be the opinion of Chasles: but a more appropriate opportunity for the examination of this question will occur hereafter.

GEOMETRICAL PROPOSITIONS.

[*Mr. James Dalmahoy, Edinburgh.*]

PROPOSITION I.

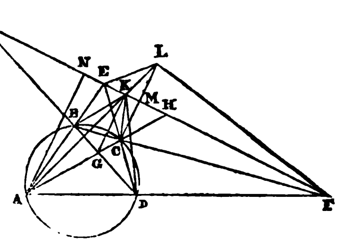
Let ABCD be a quadrilateral figure inscribed in a circle, and having its opposite sides produced to meet in E and F. Draw the diagonals AC, BD intersecting in G, and produce them to meet a line drawn through E, F in the points H and I. From the points A, C and G let fall the perpendiculars AN, CM and GK upon EF. Join A, K, and produce AK and CM to meet in L; and draw the lines CK, EL and FL.

It may be demonstrated that

$$AF.EC + AE.FC = EF\,(AK + CK).$$

For we have CH : AH : : CM : AN; also CG : AG : : KM : KN; but (Hut. II. 223) CH : AH : : CG : AG; ∴ CM : AN : : KM : KN.

Hence the triangles CKM, AKN are evidently similar, and the angles CKM, and AKN are equal. Again, the angle LKM being equal to its vertical angle AKN, is equal to CKM; consequently the right angled triangles LKM and CKM are evidently equal, and CK equal to KL, and also CM to ML. It is further evident that the triangles EML and FLM are respectively equal to ECM and FCM;



consequently, that the whole triangle ELF is equal to ECF; the side EL to EC, the side FL to FC, and the angle FLE equal to FCE equal to BCD. Hence the angle FLE is the supplement of FAE, and the quadrilateral FAEL may be inscribed in a circle; therefore

$$AF.EL + AE.FL = EF.AL.$$

But EL = EC; FL = FC, and AL = AK + CK; therefore

$$AF.EC + AE.FC = EF\,(AK + CK).$$

PROP. II.

In the same figure join the points B, K and D, K.

It may be demonstrated that

$$BF.DE + BE.DF = EF\,(BK + DK).$$

By considering that DI is harmonically divided in B and G; and that the angle EBF is supplementary to FDE, it will appear that this proposition may be demonstrated in a manner exactly similar to the former one.

Cor. It is evident that

$$EK.KF = AK.CK = BK.DK.$$

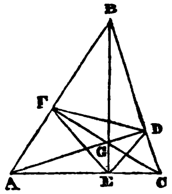
PROP. III.

Let ABC be an acute angled triangle; AD, BE and CF perpendiculars let fall from the angular points upon the opposite sides, and G their common intersection. Join the points D, E and F.

It may be demonstrated that

$$(AB + BC + CA)(DE + EF + FD) = 2(AD.CF + AD.BE + BE.CF);$$

or in an acute angled triangle, twice the sum of the rectangles under the perpendiculars let fall from the angles upon the opposite sides, is equal to the rectangle under the sum of the sides of the triangle, and the sum of the lines joining the points of intersection of the perpendiculars with the opposite sides.



For it is evident that the quadrilateral BDGF may be inscribed in a circle; hence, by Prop. II.,

$$AD.CF + AF.CD = AC(DE + EF).$$

Also, since a circle may be described through the points A, F, D, C; therefore

$$AD.CF - AF.CD = AC.DF;$$

consequently, by adding together the corresponding sides of these equations, there results

$$2AD.CF = AC(DE + EF + FD).$$

In the same manner, it may be proved that

$$2AD.BE = AB(DE + EF + FD);$$

and, $$2BE.CF = BC(DE + EF + FD).$$

Hence, by adding together these equals,

$$(AB + BC + CA)(DE + EF + FD) = 2(AD.CF + AD.BE + BE.CF).$$

PROP. IV.

Let AGC (last figure) be a triangle, obtuse angled at G; let AF, CD and GE be perpendiculars let fall from the angular points upon the opposite sides, and B their common point of intersection.

It may be demonstrated that

$$(AC + CG + GA)(DE + EF - FD) = 2(AF.CD + AF.GE + CD.GE)$$

As was before proved,

$$AF.CD = AC(DE + EF) - AD.CF.$$

Also, $$AF.CD = AD.CF - AC.DF;$$

Adding the corresponding sides of these equations, we get

$$2AF.CD = AC(DE + EF - FD.)$$

Again, since the triangle AFD is similar to AGB, and the triangle AED to AGC; also, since a circle may be described about AFGE, it follows that

$$AG.FD = AF.BG \ldots (1) \qquad AG.DE = AE.CG \ldots\ldots\ldots (2)$$

$$AG.FE = AF.GE + AE.FG \ldots\ldots\ldots\ldots\ldots (3)$$

Therefore, by adding together the corresponding sides of these equations, and by the similarity of the triangles AEB, AFC, we obtain

$$AG(DE + EF + FD) = AF.BE + AE.CF = 2AF.BE.$$

Subtracting from this twice the corresponding sides of equation (1), there results

$$AG(DE+EF-FD)=2AF.GE$$

In the same manner it may be proved that

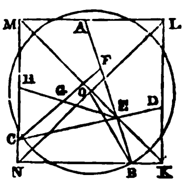
$$CG(DE+EF-FD)=2CD.GE;$$

$$\therefore (AC+CG+GA)(DE+EF-FD)=2(AF.CD+AF.GE+CD.GE).$$

The former enunciation, therefore, holds also for obtuse angled triangles, provided that the line joining the two exterior intersections of the perpendiculars with the opposite sides be considered negative.

PROP. V.

Construction of a rectilineal figure approximately equal to the area of a given circle, and probably derived from some Hindoo work on geometry.



Draw the indefinite line AB, and divide it into three equal parts AF, FE, EB. Through E, and at right angles to AB, draw the line CD: make CE equal to AE; ED equal to EB, and, consequently, CD itself equal to AB. Join the points C and F. From CF cut off FG equal to FE; and through G draw, from the point E, the line EH equal to CE. Through C and H draw the indefinite line MN; and from the points A and B let fall perpendiculars meeting it in M and N, and produced indefinitely towards L and K. Through D draw the line KL parallel to MN, and meeting the lines ML and NK in the points L and K. Further, draw the diagonals MK, LN intersecting in the point O. Join OB, and from O as a centre, with the radius OB, describe a circle BHAD.

Sch.—The foregoing construction was sent to the Madras Literary Society, I believe about the year 1826, by a gentleman of the Civil Service of that Presidency. No demonstration accompanied it, but only the enunciation that the area of the circle of which OB is the radius, is equal to that of the rectilineal figure MLKN.

Having by accident, seen this enunciation in 1829, I transmitted to the author of it, through a friend, a demonstration that the construction led to the following result.

The rectilineal figure MLKN $=$

$$\frac{18\{\sqrt{(2\sqrt{5})}+\sqrt{(\sqrt{5}+1)}\}}{7\sqrt{(2\sqrt{5})}+3\sqrt{(\sqrt{5}+1)}+2\sqrt{(\sqrt{5}-1)}}OB^2=3{\cdot}14160{}^{2+}_{3-}OB^2;$$

consequently, that MLKN was greater than the circle of which OB is the radius, the area of the latter being, as is known, between the limits $3{\cdot}14159{}^{2+}_{3-}OB^2$.

The gentleman stated in reply, that he had, some time before, discovered the enunciation to be only approximately true, and that at the moment he was engaged in drawing up an account of the construction as a specimen, I understood him to say, of a Hindoo attempt at the quadrature of the circle.

The gentleman thus alluded to was an excellent scholar in the languages of India, and may have met with the construction in some native work on geometry; he, however, died a few years after, without sending, as far as I know, any further communication on the subject.

SOLUTION OF A PROBLEM RELATING TO ATTRACTION.

[*Mr. James Anderson, Montrose.*]

In the Mecanique Celeste, Laplace has investigated the laws of attraction which render the attraction of a homogeneous spherical shell upon an external point the same as if the mass of the shell were collected at its centre; and also the law which reduces the attraction upon an internal point to zero. The latter of these problems has also been discussed by Murphy, in his Treatise on Electricity. The following solutions are different, and may perhaps not be unacceptable to the mathematical student.

Let r be the distance of the point from the centre of the shell,
u the radius of the shell, du its thickness,
θ the angle made by a moveable radius with the fixed radius drawn towards the point, and
f the distance of the point from the extremity of this moveable radius.

Then it is evident that the total attraction of the shell upon the point, whether internal or external, is

$$2\pi u^2 du \int_0^\pi \phi(f) \frac{r - u\cos\theta}{f} \sin\theta \, d\theta \;\ldots\ldots\ldots\ldots\ldots (1)$$

$\phi(f)$ denoting the unknown law of attraction.

By the geometry of the question, we have

$$f^2 = u^2 + r^2 - 2ur\cos\theta,$$

from which we deduce

$$f df = ur \sin\theta \, d\theta;$$

hence also,

$$\text{when} \quad \theta = 0, f = \pm(u - r),$$
$$\text{and when} \quad \theta = \pi, f = \pm(u + r).$$

When the point is external, r is greater than u, and when it is internal, r is less than u; hence, since f is supposed to be positive, in formula (1), the limiting values of f, in the former case, are $r + u$ and $r - u$; and in the latter, $u + r$ and $u - r$.

Eliminating θ from formula (1) it is easy to find, that when the point is external, the attraction is

$$\pi u du \int_{r-u}^{r+u} \phi(f) \frac{r^2 + f^2 - u^2}{r^2} df;$$

and when the point is internal, it is

$$\pi u du \int_{u-r}^{u+r} \phi(f) \frac{r^2 + f^2 - u^2}{r^2} df.$$

The mass of the shell, its density being 1, is $4\pi u^2 du$, and if it attracted as if its mass were collected at its centre, the attraction upon the point would be $4\pi u^2 du \phi(r)$; hence, in the first problem, $\phi(f)$ must be such that

$$\int_{r-u}^{r+u} \phi(f) \frac{r^2 + f^2 - u^2}{r^2} df = 4u\phi(r) \;\ldots\ldots\ldots\ldots\ldots (2)$$

whilst in the second, it must be such that

$$\int_{u-r}^{u+r} \phi(f)\frac{r^2+f^2-u^2}{r^2}df=0 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Differentiating equation (2) with respect to u, and keeping in view that the limiting values of f depend upon u, we obtain

$$-\frac{2u}{r^2}\int_{r-u}^{r+u} \phi(f)df+\frac{2(r-u)}{r}\phi(r-u)+\frac{2(r+u)}{r}\phi(r+u)=4\phi(r).$$

Representing the indefinite integral $\int f\phi(f)df$ by $\Psi(f)$, and reducing, there results

$$-\frac{u}{r}\{\Psi(r+u)-\Psi(r-u)\}+(r-u)\phi(r-u)+(r+u)\phi(r+u)-2r\phi(r)=0$$

This equation must be true irrespective of the value of u, and hence when it is developed in ascending powers of u, the coefficients of the different powers must be separately equal to zero. Now we find that there are no odd powers, and that the coefficient of u^{2n}, when n is any positive integer, *exclusive of zero*, is

$$\frac{2}{1\cdot 2\cdot 3\cdot\ldots\ldots 2n}\left\{-\frac{2n}{r}\Psi^{(2n-1)}r+r\phi^{(2n)}r+2n\phi^{(2n-1)}r\right\};$$

Consequently, $\Psi^{(2n-1)}r$ being equal to $\phi^{(2n-2)}r$,

$$-2n\phi^{(2n-2)}r+r^2\phi^{(2n)}r+2nr\phi^{(2n-1)}r=0\ldots\ldots\ldots\ldots(4)$$

Differentiating with respect to r, and reducing, we have

$$\phi^{(2n+1)}r+\frac{2(n+1)}{r}\phi^{(2n)}r=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and integrating,

$$\log\phi^{(2n)}r=2(n+1)\log\frac{c}{r},$$

$$\text{or }\phi^{(2n)}r=\left(\frac{c}{r}\right)^{2(n+1)},$$

from which, by two integrations, and writing C for $\frac{c^{2(n+1)}}{2n(2n+1)}$, we find

$$\phi^{(2n-2)}r=A+Br+\frac{C}{r^{2n}}.$$

This is the complete integral of (5); but it is to satisfy (5), which it does not, unless $A=0$; hence

$$\phi^{(2n-2)}r=Br+\frac{C}{r^{2n}},$$

and making $n=1$, and using new constants, we get

$$\phi(r)=B_1r+\frac{C_1}{r^2};$$

so that the law of attraction must be proportional to the distance, or inversely proportional to the square of the distance, or partly the one and partly the other.

Resuming eq. (3), and differentiating with respect to u, we get

$$-2u\int_{u-r}^{u+r}\phi(f)df+2r(u+r)\phi(u+r)+2r(u-r)\phi(u-r)=0\,..\,(6)$$

or, as before,

$$-u\{\Psi(u+r)-\Psi(u-r)\}+r(u+r)\phi(u+r)+r(u-r)\phi(u-r)=0$$

Developing this in powers of r, equating the coefficient of r^{2n+1} (there being no even powers of r) to zero, and reducing, there results

$$-u\Psi^{(2n+1)}u+(2n+1)u\phi^{(2n)}u+2n(2n+1)\phi^{(2n-1)}u=0$$

whence $$u\phi^{(2n)}u=-(2n+1)\phi^{(2n-1)}u.$$

Integrating, $$\phi^{(2n-1)}u=\left(\frac{c}{u}\right)^{2n+1},$$

and $$\phi^{(2n-2)}u=\mathrm{A}+\frac{\mathrm{C}}{u^{2n}},\text{ writing C for }-\frac{c^{2n+1}}{2n}.$$

This result satisfies (6), and is the most general equation which satisfies it; but it does not satisfy eq. (3), unless $\mathrm{A}=0$; hence

$$\phi^{(2n-2)}u=\frac{\mathrm{C}}{u^{2n}},$$ C of course being different for different values of n. When $n=1$, let C be C_1, and then

$$\phi(u)=\frac{\mathrm{C}_1}{u^2},$$

so that in this case the only law admissible is that of the inverse squares.

It is evident that the investigation and the results are the same, when the forces are of repulsion instead of attraction, the signs of the constants in the results being different. It is evident that the free electricity on an isolated sphere will be equally diffused on the surface, and it is found by observation that it is all collected on the surface, so that its total repulsion on an interior particle is zero. It therefore follows, by the last problem, that the law of electrical repulsion is that of the inverse squares.

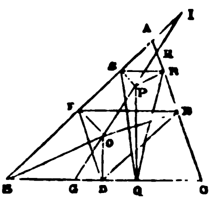
ANALYTICAL DEMONSTRATION OF A GEOMETRICAL THEOREM.

[*From a Correspondent.*]

If from any point in the plane of a triangle perpendiculars be let fall upon the sides, and the extremities of these perpendiculars be joined two and two; if the triangle thus formed be denoted by Δ', the triangle formed by joining, two and two, the points of bisection of the sides of the original triangle by Δ, the radius of the circumscribing circle by R, and the distance between the assumed point and the centre of the circumscribing circle by R$'$, then

$$\frac{\Delta'}{\Delta}=\pm\frac{\mathrm{R}^2-\mathrm{R}'^2}{\mathrm{R}^2},$$

ac[illegible] as the assumed point is *within* or *without* the circumscribed circle.



Let O be the centre of the circumscribed circle; OD, OE, OF the perpendiculars bisecting the sides, and DEF the triangle formed by joining these points. Let P be the assumed point, and QRS the triangle formed by joining the extremities of the perpendiculars from P upon the sides. Join PO and produce it both ways, if required, to meet the sides in G, H, I. Then denoting the angle PGC by a, we have

$$\begin{aligned}
&\mathrm{OD} = \mathrm{R}\cos \mathrm{A}; \ \mathrm{OE} = \mathrm{R}\cos \mathrm{B}; \ \mathrm{OF} = \mathrm{R}\cos \mathrm{C};\\
&\mathrm{PQ} = \mathrm{OD} + \mathrm{OP}\sin a = \mathrm{R}\cos \mathrm{A} + \mathrm{R}'\sin a;\\
&\mathrm{PR} = \mathrm{OE} - \mathrm{OP}\sin(a+\mathrm{C}) = \mathrm{R}\cos \mathrm{B} - \mathrm{R}'\sin(a+\mathrm{C});\\
&\mathrm{PS} = \mathrm{OF} - \mathrm{OP}\sin(a-\mathrm{B}) = \mathrm{R}\cos \mathrm{C} - \mathrm{R}'\sin(a-\mathrm{B});
\end{aligned}$$

where a, $a+\mathrm{C}$ and $a-\mathrm{B}$ are the angles, or their supplements, which the line PO produced makes with the sides of the triangle ABC.

$$\begin{aligned}
\text{Now } 2\Delta &= \mathrm{OD.OE}\sin \mathrm{C} + \mathrm{OE.OF}\sin \mathrm{A} + \mathrm{OF.OD}\sin \mathrm{B}\\
&= \mathrm{R}^2(\cos \mathrm{A}\cos \mathrm{B}\sin \mathrm{C} + \cos \mathrm{B}\cos \mathrm{C}\sin \mathrm{A} + \cos \mathrm{C}\cos \mathrm{A}\sin \mathrm{B}).
\end{aligned}$$

Consequently we have

$$\begin{aligned}
2\Delta' &= \mathrm{PQ.PR}\sin \mathrm{C} + \mathrm{PR.PS}\sin \mathrm{A} + \mathrm{PS.PQ}\sin \mathrm{B}\\
&= 2\Delta + \left\{\begin{matrix}\sin a\cos \mathrm{B}\sin \mathrm{C} - \sin(a+\mathrm{C})\cos \mathrm{A}\sin \mathrm{C} - \sin(a-\mathrm{B})\cos \mathrm{A}\sin \mathrm{B}\\ \sin a\sin \mathrm{B}\cos \mathrm{C} - \sin(a+\mathrm{C})\sin \mathrm{A}\cos \mathrm{C} - \sin(a-\mathrm{B})\sin \mathrm{A}\cos \mathrm{B}\end{matrix}\right\}\mathrm{RR}'\\
&\quad - \{\sin a\sin \mathrm{C}\sin(a+\mathrm{C}) + \sin a\sin \mathrm{B}\sin(a-\mathrm{B}) - \sin \mathrm{A}\sin(a+\mathrm{C})\sin(a-\mathrm{B})\}\mathrm{R}'^2.
\end{aligned}$$

But the coefficient of RR' is zero; for adding the vertical terms, it is equal to

$$\sin a\sin(\mathrm{B}+\mathrm{C}) - \sin(a+\mathrm{C})\sin \mathrm{B} - \sin(a-\mathrm{B})\sin \mathrm{C} = 0;$$

and the coefficient of R'^2 can be transformed into $\sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C}$ in the following manner. Since

$$\begin{aligned}
\sin a\sin \mathrm{C}\sin(a+\mathrm{C}) &= \sin^2 a\sin \mathrm{C}\cos \mathrm{C} + \sin a\cos a\sin^2 \mathrm{C},\\
\sin a\sin \mathrm{B}\sin(a-\mathrm{B}) &= \sin^2 a\sin \mathrm{B}\cos \mathrm{B} - \sin a\cos a\sin^2 \mathrm{B},\\
\sin \mathrm{A}\sin(a+\mathrm{C})\sin(a-\mathrm{B}) &= \sin^2 a\cos \mathrm{B}\cos \mathrm{C}\sin \mathrm{A} + \sin a\cos a\cos \mathrm{B}\sin \mathrm{C}\sin \mathrm{A}\\
&\quad + \sin^2 a\sin \mathrm{B}\sin \mathrm{C}\sin \mathrm{A} - \sin a\cos a\sin \mathrm{B}\cos \mathrm{C}\sin \mathrm{A}\\
&\quad - \sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C};
\end{aligned}$$

consequently the coefficient of R'^2 is

$$\begin{aligned}
&\sin a\sin \mathrm{C}\sin(a+\mathrm{C}) + \sin a\sin \mathrm{B}\sin(a-\mathrm{B}) - \sin \mathrm{A}\sin(a+\mathrm{C})\sin(a-\mathrm{B})\\
&= \sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C} + \sin^2 a\{\sin \mathrm{C}\cos \mathrm{C} + \sin \mathrm{B}\cos \mathrm{B} - \sin \mathrm{A}\cos(\mathrm{B}-\mathrm{C})\}\\
&\qquad + \sin a\cos a\{\sin^2 \mathrm{C} - \sin^2 \mathrm{B} + \sin \mathrm{A}\sin(\mathrm{B}-\mathrm{C})\}\\
&= \sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C} + \sin^2 a\{\tfrac{1}{2}\sin 2\mathrm{C} + \tfrac{1}{2}\sin 2\mathrm{B} - \sin(\mathrm{B}+\mathrm{C})\cos(\mathrm{B}-\mathrm{C})\}\\
&\qquad + \sin a\cos a\{\sin^2 \mathrm{C} - \sin^2 \mathrm{B} + \sin(\mathrm{B}+\mathrm{C})\sin(\mathrm{B}-\mathrm{C})\}\\
&= \sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C},
\end{aligned}$$

the sum of the terms in each of the bracketed expressions being zero; therefore the expression for twice the area of the triangle QRS is

$$2\Delta' = 2\Delta - \mathrm{R}'^2\sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C}.$$

But by Art. XXIX, cor. 8, Horæ Geometricæ, Lady's Diary for 1842,

$$R^2 \sin A \sin B \sin C = 2\Delta;$$

consequently, by substitution, and reducing, we finally obtain

$$\frac{\Delta'}{\Delta} = \frac{R^2 - R'^2}{R^2}.$$

In the same manner, when the point P is *without* the circumscribing circle, we get

$$\frac{\Delta'}{\Delta} = \frac{R'^2 - R^2}{R^2};$$

and hence according as the assumed point is *within* or *without* the circumscribing circle, we have

$$\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{R^2}.$$

Cor. 1.—If lines PQ$'$, PR$'$, PS$'$ be drawn, making equal angles on the same side of the lines with the sides of the triangle, then the angles contained between these lines, two and two, will remain unchanged, while their lengths will be increased in the same proportion. For denote the angle which either of the lines makes with the side of the triangle by β; then we have

$$PQ' = PQ \sec \beta,\ PR' = PR \sec \beta,\ PS' = PS \sec \beta,$$

and consequently the area of the triangle $Q'R'S'$, formed by joining the extremities of these lines, will be

$$Q'R'S' = QRS \sec^2 \beta = \Delta' \sec^2 \beta;$$

and if the triangle $Q'R'S'$ be denoted by Δ_1, we shall have generally

$$\frac{\Delta_1}{\Delta} = \pm \frac{R^2 - R'^2}{R^2} \operatorname{cosec}^2 \beta;$$

the $+$ corresponding to the case R greater than R$'$, and the $-$ when R less than R$'$.

Cor. 2.—Let P be any point in the circumference of a circle concentric with the circumscribing circle, and let the angle formed by tangents drawn from any point in the outer circle to the inner be denoted by 2θ; then

$$\frac{\Delta'}{\Delta} = \cos^2\theta, \text{ or } \frac{\Delta'}{\Delta} = \cot^2 \theta,$$

according as the point P is *within* or *without* the circumscribing circle, or which is the same thing, according as R is greater or less than R$'$.

For we have evidently, in the former case,

$$R' = R \sin \theta;\ \therefore\ R^2 - R'^2 = R^2 \cos^2 \theta;$$

and, in the latter case,

$$R' = R \operatorname{cosec} \theta;\ \therefore\ R'^2 - R^2 = R^2 \cot^2 \theta;$$

Hence $$\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{R^2} = \cos^2 \theta, \text{ or } \cot^2 \theta.$$

Cor. 3.—When $R = 2R'$; then we have

$$\frac{\Delta'}{\Delta} = \frac{R^2 - R'^2}{R^2} = \tfrac{3}{4}, \text{ or } 3\Delta = 4\Delta';$$

hence, $$\cos^2 \theta = \frac{\Delta'}{\Delta} = \tfrac{3}{4};\ \therefore\ \theta = 30°, \text{ or } 2\theta = 60°.$$

Scholium.—It appears from Cor. 2, that Δ' varies as $\cos^2 \theta$ or as $\cot^2 \theta$, Δ being constant.

By means of the preceding proposition may be demonstrated the well known theorem—"If from any point in the circumference of a circle described about a triangle, perpendiculars be let fall upon the sides, then the feet of these perpendiculars will be in the same straight line." For a synthetical demonstration see "Mathematical Companion," or "Bland's Problems."

When in the formula, $R = R'$, the point P falls on the circumference of the circle described about the triangle ABC, and the Δ' vanishes, or, which is the same thing, the angular points of it, which are then the extremities of the perpendiculars let fall upon the sides, are in the same straight line.*

By an application of the above theorem—"If from any point in the circumference of a circle, etc.," we may demonstrate the following most elegant theorem deduced by C. F. A. Jacobi from different and less simple principles.

THEOREM.

If perpendiculars be let fall from the angular points of a triangle upon the three sides respectively, and from the extremity of any one of these perpendiculars lines be drawn at right angles to the other two perpendiculars and the other two sides; the extremities of these four perpendiculars are in one straight line, which is parallel to the line joining the extremity of the other two perpendiculars drawn from the angular points to the sides.

Let ABC be the triangle (the student will easily draw the figure) AD, BE, CF, the perpendiculars from the angular points upon the sides, then from the point E, the extremity of one of these lines, let the perpendiculars EP_1, EP_2, EP_3, EP_4 be drawn to the lines CB, CF, AD, AB respectively; the points P_1, P_2, P_3, P_4 shall be in the same straight line, which shall be parallel to DF.

Let G be the intersection of the perpendiculars AD, BE, CF, and let circles be described about the triangles AFG, BCF; then these circles will pass through E (see Bland's Geo. Problems, section 7, prob. 40.)

Now, since the circle passing through E, C, B is about the triangle BCF, and EP_1, EP_2, EP_4, are the perpendiculars upon its sides, it follows that P_1, P_2, P_4 are in the same straight line.

Again, since the circle passing through A, E, G is about the triangle AFG, and EP_2, EP_3, EP_4 are perpendiculars upon its sides, it follows that P_2, P_3, P_4 are in the same straight line.

Hence P_1, P_2, P_3, P_4 are all in the same straight line.

Also, we have

$$P_4F : FB :: EG : GB : P_1D : DB,$$

therefore the line $P_1\ P_2\ P_3\ P_4$ is parallel to DF.

Durham, Oct. 12th, 1843. W. F.

* This method of proving three points in the same straight line, is very ingenious, and applicable, probably, in many other cases. It is desirable, however, that the remarkable theorem on which it is founded, should be established by pure geometry. This will form a good exercise for the student. We shall be glad to give its investigation, in this way, in some future number of the Mathematician, should such be furnished by any of our correspondents.
EDS.

ON HORNER'S SYNTHETIC DIVISION.

The importance of this method in its applications to developement in series of fractional expressions,—to the numerical solution of algebraical equations,—to the transformation of expressions into factorials,—to the formation of recurring series, and to several other purposes,—renders it an extremely valuable instrument in the hands of the analyst. Mr. Horner himself has given a very elegant proof of it (see Ladies' Diary, 1838); but as that is not so simple and intelligible as could be desired for the purposes of elementary study, it will be useful to point out a proof, the principle of which brings it within the reach of the student at his first entrance on the use of the method.

1. The method of employing detached coefficients is suggested at once by the corresponding application of it in common arithmetic, the numbers put down in each "place" being only the coefficients of corresponding powers of 10. Examples of the use of this abbreviation are to be found in several works, and the use of them may be assumed as known.

2. It is to be borne in mind, too, that in analogy to addition in common arithmetic, the *sum* of several given numbers is expressed as equal to another stated number, not by means of the sign of equality ($=$) written between them horizontally, but by a line separating the two parts of the equality. In fact, we substitute the *vertical* arrangement of the quantities, as in arithmetic, for the *horizontal*, ordinarily employed in algebra. Thus, $135+612+518=1265$, is the algebraical form: but the arithmetical is

$$\begin{array}{r} 135 \\ 612 \\ 518 \\ \hline 1265 \end{array} \qquad \text{or} \qquad \begin{array}{c} 135 \\ 612 \\ 518 \\ \| \\ 1265 \end{array}$$

All that we require, therefore, is this change of arrangement of the work, the line under a column being understood to represent the symbol $=$.

3. Let then any two polynomes be given, one of which is to be divided by the other, the coefficient of the first term of this being unity; and let them and the resulting quotient be denoted by

$$\mathrm{A}x^m+\mathrm{B}x^{m-1}+\mathrm{C}x^{m-2}+\ldots\ldots \quad (\text{the } dividend) = f(x),$$

$$x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots \quad (\text{the } divisor) = f_1(x),$$

$$\mathrm{A}x^{m-n}+\mathrm{A}_1x^{m-n-1}+\mathrm{A}_2x^{m-n-2}+\ldots \quad (\text{the } quotient) = f_2(x);$$

the last of which will, in most cases, proceed with negative powers of x *ad infinitum* when the arrangement of the terms is in descending powers, and in positive powers of x when ascending.

4. The rule given by Mr. Horner is:—

(*a*) Write the coefficients of the dividend in a horizontal line with their proper signs.

(*b*) Draw a vertical line to the left of the first coefficient of the dividend, and in a vertical column set down the coefficients of the divisor, with all their signs, except the first, changed.

(*c*) Draw beneath the last of this column of coefficients a horizontal line, to serve as the sign of equality of the columns of coefficients, to come in above it, with the single coefficients below it, in the respective columns, as in ordinary arithmetical addition.

[Should any terms of either divisor or dividend be wanting, their places to be supplied with ciphers.]

(*d*) Bring down the first coefficient of the dividend as the first of the quotient.

(*e*) Multiply each term of the vertically arranged divisor (except the first or unity) by the first quotient figure (found in *d*), and place the products on the same horizontal lines with those multiplicands, but each successive one a step further to the right than the preceding one; and thus form the first oblique column.

(*f*) Add the coefficients of the second column, which will give that of the second term of the quotient.

(*g*) With the coefficient as a multiplier of the terms of the dividend (as written in *b*) form the second oblique column, in juxtaposition with the first.

(*h*) Add the terms now in the next vertical column, which gives the coefficient of the next term of the quotient.

(*k*) With this new coefficient form the next oblique column, and then find the next coefficient of the quotient as before.

(*l*) Proceed in the same course of operations either till the work terminates or as many terms as may be required for the purpose in view are obtained.

(*m*) Prefix the coefficients thus obtained, with their proper signs, to x^{m-n}, x^{m-n-1}, etc.

Such is the verbal rule: the work itself will stand as follows.

$$
\begin{array}{r|lllll}
1 & A + & B + & C + & D + \ldots\ldots & \\
-a_1 & & -Aa_1 - & A_1a_1 - & A_2a_1 - \ldots\ldots & \\
-a_2 & & & -Aa_2 - & A_1a_2 - & A_2a_2 - \ldots \\
-a_3 & & & & -Aa_3 - & A_1a_3 - A_2a_3 - \ldots\ldots \\
\vdots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\hline
 & A + & A_1 + & A_2 + & A_3 + & \ldots\ldots
\end{array}
\quad (1)
$$

In which $A_1 = B - Aa_1$, $A_2 = C - A_1a_1 - Aa_2$, etc., and the oblique columns are respectively

$$
\begin{array}{l}
-Aa_1 - Aa_2 - Aa_3 - \ldots \\
-A_1a_1 - A_1a_2 - A_1a_3 - \ldots \\
-A_2a_1 - A_2a_2 - A_2a_3 - \ldots
\end{array}
$$

The truth of the rule is thus proved:

(*A*) Considering the work above to represent an arithmetic equation, transpose all the terms of the oblique columns to the other side of the sign

of equality, changing at the same time their signs: then the work will stand as below.

$$\begin{array}{llllll}
A + A_1 + A_2 + A_3 \ldots\ldots \\
\quad + Aa_1 + A_1a_1 + A_2a_1 + \ldots\ldots \\
\qquad + A\,a_2 + A_1a_2 + A_2a_2 + \ldots\ldots \\
\qquad\quad + A\,a_3 + A_1a_3 + A_2a_3 + \ldots\ldots \\
\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\hline
A + B + C + D + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)
\end{array}$$

where the horizontal line still expresses the sign of equality.

(*B*) From the mutual relations of the divisor, dividend, and quotient, we shall have, if A, A_1, A_2, *etc.*, be correctly given by this process,

$$f_1(x) \cdot f_2(x) = f(x).$$

Or, if we put down the multiplication at length (in detached coefficients)

$$\begin{array}{l}
f_2(x) = A + A_1 + A_2 + A_3 + \ldots \\
f_1(x) = 1 + a_1 + a_2 + a_3 + \ldots \\
\hline
\quad A + A_1 + A_2 + A_3 + \ldots \\
\qquad + Aa_1 + A_1a_1 + A_2a_1 + \ldots \\
\qquad\quad + A\,a_2 + A_1a_2 + A_2a_2 + \ldots \\
\qquad\qquad + A\,a_3 + A_1a_3 + A_2a_3 + \ldots \\
\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots \\
\hline
f(x) = A + B + C + D + \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)
\end{array}$$

(*C*) It will be seen at once that (2) and (3) are identical: also, the only difference between the result of the prescribed operation, which gives equation (1), and that just performed, which gives equation (3), is:—that in the former case we determine A, A_1, A_2, *etc.*, and in the latter we find A, B, C, *etc.* The equation which gives the one set or other of coefficients is only transformed by transposition from the one side of the sign of equality to the other: wherefore, since both operations lead to the same result, and one (the multiplication) is known to be true, the other must also be true.

Jan. 8, 1844. D. V. S.

ON THE STRAIGHT LINE OF QUICKEST DESCENT.

[*From a Correspondent.*]

To determine the straight line of quickest descent from a given point to a given curve in the same vertical plane.

Let the equation of the curve be $F(x,y)=0$ ....(1)

the given point .... (a,b)

the required point............ (x,y)

the required line L

t[illegible]tion of L to the horizon i.

Then

$$L = \sqrt{\{(a-x)^2 + (b-y)^2\}}$$

and $\sin i = \dfrac{b-y}{\sqrt{\{(a-x)^2+(b-y)^2\}}}$

Now $L = \frac{1}{2}gt^2 \sin i$; $\therefore \dfrac{L}{\sin i} = \frac{1}{2}gt^2 =$ minimum,

that is, $\dfrac{(a-x)^2}{b-y} + b - y =$ minimum;

$$\therefore \left(\frac{a-x}{b-y}\right)\frac{dy}{dx} - 2\left(\frac{a-x}{b-y}\right) - \frac{dy}{dx} = 0,$$

or dividing by $\dfrac{dy}{dx}$

$$\left(\frac{a-x}{b-y}\right)^2 - 2\left(\frac{a-x}{b-y}\right)\frac{dx}{dy} - 1 = 0:$$

or dividing by $\left(\dfrac{a-x}{b-y}\right)^2$, and changing signs,

$$\left(\frac{b-y}{a-x}\right)^2 - 2\left(-\frac{dx}{dy}\right)\left(\frac{b-y}{a-x}\right) - 1 = 0 \;\ldots.(2)$$

where $-\dfrac{dx}{dy}$ is the tangent of the inclination of the normal to the axis of x, supposed horizontal, and $\dfrac{b-y}{a-x}$, the tangent of the inclination i of L.

Now when a straight line bisects the angle made by two others, the tangent k of its inclination—the tangents of the inclinations of the other lines being a_1, a_2—is given by the following condition (Davies's Hutton, vol. ii., p. 262.)

$$k^2 - 2\frac{a_1a_2 - 1}{a_1 + a_2}k - 1 = 0.$$

If one of the two proposed lines be parallel to the axis of y, one of the constants a_1, a_2 must be $\frac{1}{0}$; and then the condition becomes

$$k^2 - 2a_1k - 1 = 0.$$

This is obviously identical with (2); showing that L bisects the angle between the normal and vertical ordinate. Hence *whatever be the curve, the straight line of quickest descent to it, from a given point, always bisects the angle between the normal and vertical ordinate.*

From (1) we get

$$\frac{dx}{dy} = f(x,y) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Hence eliminating $\dfrac{dx}{dy}$ from (2) by means of (3), the resulting equation in x, y, when combined with (1), will determine the point (x, y) sought.

Belfast, January, 1844. Y.

PROPERTIES OF THE PARABOLA.

THEOREM.

A polygon of any number of sides being inscribed in a parabola, let its angular points be denoted by $p_1, p_2, p_3, \ldots p_n$ and parabolic segments upon the chords $p_1p_2, p_2p_3, p_3p_4 \ldots p_np_1$ by $s_1, s_2, s_3 \ldots s_n$: then there exists the relation

$$s_n^{\frac{1}{3}} = s_1^{\frac{1}{3}} + s_2^{\frac{1}{3}} + s_3^{\frac{1}{3}} + \ldots\ldots\ldots + s_{n-1}^{\frac{1}{3}}$$

For let $x_1y_1, x_2y_2, x_3y_3 \ldots x_ny_n$ be the co-ordinates of $p_1, p_2 \ldots p_n$, referred to the *tangent at the vertex and axis of the parabola* as axes of x and y, the equation of the parabola in this case being

$$py = x^2.$$

Then the expression for the first segment is

$$s_1 = \tfrac{1}{2}(y_2 + y_1)(x_2 - x_1) - \int_{x_1}^{x_2} y\,dx$$

$$= \frac{(x_2^2 + x_1^2)(x_2 - x_1)}{2p} - \frac{x_2^3 - x_1^3}{3p}$$

$$= \frac{1}{6p}(x_2 - x_1)^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\therefore x_2 - x_1 = (6p)^{\frac{1}{3}} \cdot s_1^{\frac{1}{3}}$$

Similarly

$$x_3 - x_2 = (6p)^{\frac{1}{3}} \cdot s_2^{\frac{1}{3}}$$

$$x_4 - x_3 = (6p)^{\frac{1}{3}} \cdot s_3^{\frac{1}{3}}$$

$$\ldots\ldots\ldots \quad \ldots\ldots$$

$$x_n - x_{n-1} = (6p)^{\frac{1}{3}} \cdot s_{n-1}^{\frac{1}{3}}$$

Hence, by addition

$$x_n - x_1 = (6p)^{\frac{1}{3}} \left\{ s_1^{\frac{1}{3}} + s_2^{\frac{1}{3}} + s_3^{\frac{1}{3}} + \ldots + s_{n-1}^{\frac{1}{3}} \right\} \ldots\ldots \quad (2)$$

But by a process similar to that for finding (1), we also get

$$x_n - x_1 = (6p)^{\frac{1}{3}} \cdot s_n^{\frac{1}{3}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$\therefore$ (2, 3)

$$s_n^{\frac{1}{3}} = s_1^{\frac{1}{3}} + s_2^{\frac{1}{3}} + s_3^{\frac{1}{3}} + \ldots\ldots\ldots + s_{n-1}^{\frac{1}{3}}.$$

T[illegible]orem is known to some of the continental mathematicians. Indepe[illegible] its extreme elegance, it cannot fail to be highly interesting to

the mathematical student on account of the many elegant properties which are deducible from it. Some of these are given below as corollaries. It must be borne in mind that the co-ordinates $x_1\, y_1,\ x_2\, y_2,$ *etc.* are taken in the order of their magnitude, of which $x_n\, y_n$ are the greatest.

Corollary 1.

If $\Delta_1,\ \Delta_2,\ \Delta_3 \ldots . \Delta_n$, are the triangles formed by the chords $p_1\, p_2,\ p_2\, p_3 \ldots . p_n\, p_1$, and the tangents at their extremities; then

$$\Delta_n^{\frac{1}{3}} = \Delta_1^{\frac{1}{3}} + \Delta_2^{\frac{1}{3}} + \Delta_3^{\frac{1}{3}} + \ldots\ldots\ldots + \Delta_{n-1}^{\frac{1}{3}}$$

This is deduced directly from the general theorem just given, and the well known property that "a parabolic segment cut off by any chord is two-thirds of the triangle having that chord for its base, and the tangents at the extremities of the chord for its sides."

Corollary 2.

Let there be *three* chords of contact, so that the tangents ABD, ACF, BEC (the figure is easily conceived) at their extremities D, F, E, may form a triangle ABC; then of the triangles thus formed by the tangents and the chords of contact, viz. ADF, BED, CEF, we have the relation

$$\mathrm{ADF}^{\frac{1}{3}} = \mathrm{BED}^{\frac{1}{3}} + \mathrm{CEF}^{\frac{1}{3}}$$

This follows at once from Cor. 1.

Corollary 3.

Of the triangles formed as in the last Cor., we have also the relation

$$\mathrm{ABC}^3 = \mathrm{ADF} . \mathrm{BED} . \mathrm{CEF}$$

For, from Cor. 2 we get

$$\mathrm{ADF} = \mathrm{BED} + \mathrm{CEF} + 3\,\mathrm{BED}^{\frac{1}{3}} . \mathrm{CEF}^{\frac{1}{3}}(\mathrm{BED}^{\frac{1}{3}} + \mathrm{CEF}^{\frac{1}{3}})$$

$$= \mathrm{BED} + \mathrm{CEF} + 3\,\mathrm{BED}^{\frac{1}{3}} . \mathrm{CEF}^{\frac{1}{3}} . \mathrm{ADF}^{\frac{1}{3}}$$

$$\therefore \mathrm{ADF} - (\mathrm{BED} + \mathrm{CEF}) = 3\,\mathrm{BED}^{\frac{1}{3}} . \mathrm{CEF}^{\frac{1}{3}} . \mathrm{ADF}^{\frac{1}{3}}$$

Now, the expression on the left, is evidently equal to the triangle ABC, together with that formed by the chords DE, EF, FD; and as the latter triangle is known to be double the former, we have

$$3\mathrm{ABC} = 3\mathrm{BED}^{\frac{1}{3}} . \mathrm{CEF}^{\frac{1}{3}} . \mathrm{ADF}^{\frac{1}{3}},$$

or, $\quad \mathrm{ABC}^3 = \mathrm{BED} . \mathrm{CEF} . \mathrm{ADF}.$

This curious property may also be proved by known properties of the parabola (*Wallace's Conic Sections*, p. 170.) thus:

$$\mathrm{BA.AC} = \mathrm{BD.CF}$$
$$\mathrm{AB.BC} = \mathrm{AD.CE}$$
$$AC.CB = \mathrm{AF.BE}$$

Hence, by multiplication,

$$BA.AC.AB.BC.AC.CB = BD.BE.CF.CE.AD.AF \ldots\ldots\ldots (\alpha)$$

And since

$$\sin A.\ \sin B.\ \sin C = \sin A.\ \sin EBD.\ \sin ECF \ldots\ldots\ldots (\beta)$$

the enunciated property follows at once from (α, β).

Corollary 4.

Let us now suppose two other parabolas constructed, to which the sides AB, AC, and the other sides produced, respectively, are tangents; and let D_1, E_1, F_1 be the points of contact in AB, and CB, CA produced; also F_2, D_2, E_2 those in AC, and BA, BC produced: then

$$AD.AF.BD.BE.CE.CF = AD_1.AF_1.BD_1.BE_1.CE_1.CF_1$$
$$= AD_2.AF_2.BD_2.BE_2.CE_2CF_2$$

This remarkable property may be enunciated thus:

If the angular points of a triangle be the points of intersection of the eighteen tangents to the three parabolas which touch the sides and the sides produced of that triangle; then the products of the respective sets of tangents are equal.

It is evident from Cor. 3, that the products of the respective sets of *triangles* are equal; and since

$$\sin A = \sin D_1AF_1 = \sin D_2AF_2$$
$$\sin B = \sin EBD = \sin E_1BD_1$$
$$\sin C = \sin ECF = \sin E_2CF_2$$

the stated equalities are obtained from the relations of the triangles, by substituting the expressions for the triangles and reducing the results.

Corollary 5.

If the triangle ABC be *constant*, then the sets of products, as in the last Cor., are also constant, however the points of contact may vary.

Corollary 6.

If the chord of a parabola be divided into $(n-1)$ equal parts, and $(n-1)$ lines be drawn through these points parallel to the axis of the parabola, then the intersections of these lines with the curve being joined, form with the curve, $(n-1)$ *equal segments*.

For we have evidently

$$x_2 - x_1 = x_3 - x_2 = x_4 - x_3 \ldots\ldots\ldots = x_n - x_{n-1}$$

$\therefore\ s_1 = s_2 = s_3 \ldots\ldots\ldots = s_{n-1}$ by the expressions which immediately follow (1).

Corollary 7.

The line which joins the extremities of the chord equally divided, forms, with the curve, a parabolic segment which has to any of the equal segments formed as in the last Cor., the relation

$$s_n = (n-1)^3 s_1$$

le general theorem gives in this case,

This pendentl

$$s_n^{\frac{1}{3}} = (n-1)s_1^{\frac{1}{3}}\ \therefore\ s_n = (n-1)^3 s_1$$

Corollary 8.

Lines drawn as in the last Cor., determine also a polygon P of n sides, which has to the segment (s_n) on the chord equally divided, the relation

$$s_n = \frac{(n-1)^2}{(n-1)^2-1} \text{P}.$$

The segment s_n consists of the inscribed polygon P, and the sum of the remaining segments, so that we have

$$s_n = \text{P} + (n-1)s_1.$$

Multiply by $(n-1)^2$, and we get

$$s_n(n-1)^2 = \text{P}(n-1)^2 + s_1(n-1)^3$$
$$= \text{P}(n-1)^2 + s_n, \text{ by Cor. 7};$$
$$\therefore \quad s_n = \frac{(n-1)^2}{(n-1)^2-1} \text{P}.$$

Corollary 9.

When $(n-1)=2$, then P is a triangle, and we have

$$s_3 = \frac{4}{3} \text{P}, \text{ a well known property of the parabola.}$$

Corollary 10.

Let $p_1, p_2, p_3 \ldots . p_n$ be the angular points of a polygon of n sides inscribed in a parabola, such that *equal parabolic segments* are cut off by the chords $p_1p_2, p_2p_3, \ldots . p_{n-1}\, p_n$, and rectilineal figures of *three*, *four*, etc., sides formed, by joining p_1p_3, p_1p_4, *etc.*; then these figures will be as

$$1,\ 4,\ 10,\ 20, \ldots\ldots \tfrac{1}{6}n(n+1)(n+2).$$

For in the expression $s_n = (n-1)^3 s_1$, deduced in Cor. 7, substitute for n, the values 3, 4, 5, etc., and we have

$$\begin{array}{llllll}
s_3 = 8s_1 & \therefore & \text{P}_3 = s_3 - 2s_1 = & 8s_1 - 2s_1 = & 6s_1 = 6s_1.1 \\
s_4 = 27s_1 & \therefore & \text{P}_4 = s_4 - 3s_1 = & 27s_1 - 3s_1 = & 24s_1 = 6s_1.4 \\
s_5 = 64s_1 & \therefore & \text{P}_5 = s_5 - 4s_1 = & 64s_1 - 4s_1 = & 60s_1 = 6s_1.10 \\
s_6 = 125s_1 & \therefore & \text{P}_6 = s_6 - 5s_1 = & 125s_1 - 5s_1 = & 120s_1 = 6s_1.20 \\
\ldots\ldots & & \ldots\ldots & & \ldots\ldots \\
s_n = (n-1)^3 s_1 & \therefore & \text{P}_n = s_n - (n-1)s_1 = & (n-1)^3 s_1 - (n-1)s_1 & \\
& & = s_1 n(n-1)(n-2) = & 6s_1.\tfrac{1}{6}n(n-1)(n-2). &
\end{array}$$

Now since the series begins at the *third* term, write $(n-2)$ for n, etc., and the last result becomes $6s_1.\frac{1}{6}n\,(n+1)(n+2)$. Hence the truth of the Cor.

Corollary 11.

Lines being drawn as in the last Cor., then the triangles which have the chords p_2p_3, p_3p_4, p_4p_5.... for their bases, and the common vertex p_1, are as

$$1,\ \ 3,\ \ 6,\ldots\ldots\ldots\frac{n}{2}(n+1),$$

where n is the number of triangles.

In this case we have by Cor. 10,

$$\begin{aligned}
\Delta_1 &= P_3 = 6s_1.1\\
\Delta_2 &= P_4 - P_3 = 6s_1(4-1) = 6s_1.3\\
\Delta_3 &= P_5 - P_4 = 6s_1(10-4) = 6s_1.6\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\Delta_n &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 6s_1.\frac{n}{2}(n+1).
\end{aligned}$$

The results obtained in the last two Corollaries are very remarkable. The expressions $\frac{n}{2}(n+1)$ and $\frac{n}{6}(n+1)(n+2)$, are the same as those for the n^{th} *course*, and *sum of n courses*, of a triangular pile of balls.

January 20, 1844. S. F.

ON THE TRANSFORMATION OF ALGEBRAIC EQUATIONS.

[*James Cockle, B.A., Trin. Coll. Cam., of the Middle Temple.*]

The results given here are of greater generality and conciseness than the ordinary ones, and are obtained by a uniform process, which I shall hereafter more fully develope and extend. The reader will observe that the conclusions arrived at by Sir W. R. Hamilton, in the 6th Report of the British Association, are only true for that form of the auxiliary equation which is used by Mr. Jerrard, and do not extend to the case in which $f(y)$ is written for y therein: — for instance, if $f(y) = y^2$, a biquadratic may be reduced to the binomial form, and if $f(y) = \frac{1}{y}$ the auxiliary equation is, in the case of a cubic, materially simplified in its form. By those familiar with the properties of the roots of unity, the multiplications indicated in this paper will be performed by inspection merely.

Let $p_1\, p_2 \ldots p_n$ be the coefficients of the equation whose roots are the n quantities $v_1\, v_2 \ldots v_n$, and, a being one of the n^{th} roots of unity, let those of the expression

$$v_1 + av_2 + a^2v_3 + \ldots\ldots + a^{n-1}v_n$$

which can be formed by simultaneously changing the coefficients of *all* the quantities $v_2, v_3, \ldots v_n$ be denoted by $\phi_1(v)\ \phi_2(v)\ldots\phi_{n-1}(v)$, then, these functions are $n-1$ in number; let $\pi(v)$ represent their product. Also let $\phi_1(v^\lambda)$.. &c., and $\pi(v^\lambda)$ denote the values of the above expressions when we change v_1 into v_1^λ, v_2 into v_2^λ, &c., and let Λ be a quantity into the composition of which v does not enter. Then

1. When $n=2$, take $a=1$, then $\phi_1(v)=v_1+v_2=\pi(v)$. If $\pi(v)=0$, the equation in v is

$$v^2+A=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

2. When $n=3$, if a be one of the imaginary values of $(1)^{\frac{1}{3}}$, we have (since $\phi_1(v)=v_1+av_2+a^2v_3$ and $\phi_2(v)=v_1+a^2v_2+av_3$),

$$\pi(v)=\Sigma(v^2_1)-\Sigma(v_1v_2)=p^2_1-3p_2.$$

Hence, if $\pi(v)=0$, the equation in v is $\left(v+\frac{p_1}{3}\right)^3+A=0$,

$$\text{or,} \qquad Y^3+A=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

3. When $n=4$, take $a=-1$, then

$$\pi(v)=\phi_1(v)\times\phi_2(v)\times\phi_3(v)=\Sigma(v^3_1)-\Sigma(v^2_1v_2)+2\Sigma(v_1v_2v_3)$$
$$=-8p_3+4p_1p_2-p^3_1,$$

which shows that when $\pi(v)=0$, the equation in v is

$$\{v^2+\frac{p_1}{2}.\ v+\frac{1}{8}(4p_2-p^2_1)\}^2+A=0, \text{ or}$$
$$Y^2+A=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and the further relation $3p^2_1=8p_2$ is necessary before the equation can be reduced to

$$Y^4+A=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Let x be the root of any given equation, and

$$v=\Lambda x^\lambda+\Lambda' x^{\lambda'},$$

and for convenience let $z=\frac{\Lambda}{\Lambda'}$, then,

4. In quadratics $\pi(v)=0$ gives $z\pi(x^\lambda)+\pi(x^{\lambda'})=0$, or, denoting by an accent the change of λ' into λ

$$z.\pi(x^\lambda)+\pi'(x^\lambda)=0 \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The object of this last reduction will be seen immediately. It is scarcely necessary to observe, that the functions indicated by π are *symmetric*, and that the simplest transformation of a quadratic to the binomial form is when

$$\Lambda'=1 \qquad \lambda=0 \qquad \text{and } \lambda'=1,$$

and, therefore, $\Lambda=z=\frac{a}{2}$, the ordinary method.

5. In cubics $\pi(v)=0$ gives $\phi_1(v)=0$ and $\phi_2(v)=0$, which are respectively equivalent to

$$z.\phi_1(x^\lambda)+\phi_1(x^{\lambda'})=0 \text{ and } z.\phi_2(x^\lambda)+\phi_2(x\lambda')=0,$$

which equations being multiplied together and substitution made for $\phi_1(x^{\lambda})$, &c., give for determining z, the quadratic

$$z^2\{\Sigma(x_1^{2\lambda})-\Sigma(x_1x_2)^{\lambda}\}+z\{\Sigma(x_1^{\lambda+\lambda'})-\Sigma(x_1^{\lambda}x_2^{\lambda'})\}+\Sigma(x_1^{2\lambda'})-\Sigma(x_1x_2)^{\lambda'}=0$$

the coefficients of which are symmetrical functions of the roots of the given equation. For the sake of uniformity and symmetry we may, as in the last case, express this equation by

$$z^2.\pi(x^{\lambda})+z.\pi'(x^{\lambda})+\pi''(x^{\lambda})=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where the number of accents denotes the number of λ''s which have been changed into λ's. If we take $\lambda=0$ the two first terms of (6) vanish, if $\lambda'=0$ the two last terms vanish, and in neither case do we obtain any result that can avail us in transforming a cubic to the binomial form. The simplest assumption for that purpose is $v=\Lambda x+x^2$.

6. For biquadratics we have the additional equation $\phi_3(v)=0$, or $z.\phi_3(x^{\lambda})+\phi_3(x^{\lambda'})=0$ and multiplying and substituting as in the last case we obtain the final cubic

$$z^3.\pi(x^{\lambda})+z^2.\pi'(x^{\lambda})+z.\pi''(x^{\lambda})+\pi'''(x^{\lambda})=0\ldots\ldots\ldots\ldots(7)$$

for reducing the given equation to the form (3), but neither λ nor λ' must equal zero, and the simplest assumption is $v=\Lambda x+x^2$. It would not be difficult to show, that, whatever be the number of terms contained in the expression for v, it is impossible to satisfy the relation $3p_1^2=8p_2$, and so destroy the three middle terms of a biquadratic (see Sir W. Hamilton's paper before referred to). But (3) may be reduced to the form $(v'^2+a^2)^2+A=0$ and from that to $y^4+B=0$, hence it might seem that the above difficulty was obviated, but in this latter case the auxiliary equation will be found to be

$$y^2+Mx^{\mu}+M'x^{\mu'}+\&c.=0,$$

which is different from Mr. Jerrard's.

The importance of a change in the form of the auxiliary equation is evident from the fact that if we take

$$\frac{1}{v}=\Lambda x^{\lambda}+\Lambda' x^{\lambda'}$$

and (for cubics) substitute this value of v in $\phi_1(v)=0$ and $\phi_2(v)=0$, we shall, on clearing those equations of fractions, obtain two simple equations, which, being multiplied together, give a quadratic *none of the coefficients of which vanish when* $\lambda=0$, and which when $\lambda=0$ and $\lambda'=1$, becomes

$$z^2\{\Sigma(x^2_1)-\Sigma(x_1x_2)\}+z\{6x_1x_2x_3-\Sigma(x^2_1x_2)\}+\Sigma(x^2_1x^2_2)-\Sigma(x^2_1x_2x_3)=0$$

This last equation, on substituting for the symmetrical functions contained in it, their values in terms of the coefficients of the given equation, is identical with the reducing equation which I have before* arrived at in the solution of a cubic with all its terms.

Temple, *26th Jan.*, 1844.

* See Cambridge Mathematical Journal, vols. II and III, the last edition of Hind's Algebra, &c.

DEMONSTRATION OF THE FOUR FUNDAMENTAL FORMULAS OF TRIGONOMETRY.

[*From a Correspondent.*]

The formulas for the solution of the several cases of *spherical triangles* are as follow; *viz.*

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

Whoever attends to the process by which these are deduced, will perceive, not only that the investigation is entirely independent of all the truths of trigonometry, except those immediately implied in the definitions, but that they universally hold for any three arcs a, b, c, on the sphere passing through three points A, B, C, even though two of these arcs, as a, b, unite continuously, forming with c a *line* instead of a *triangle*.

Suppose now that the angle A continually diminishes, at length becoming 0, the arc b then coming into coincidence with one portion of c, and consequently a coinciding with the remaining portion: then

$$a = c - b \text{ and } \cos A = 1,$$

and consequently, from the first of the above formulas

$$\cos (c - b) = \cos b \cos c + \sin b \sin c.$$

Again: suppose the angle A to continually increase till it become 180°, the arc b then becoming the continuation of c, and therefore a the sum of both: then we have

$$a = b + c \text{ and } \cos A = -1,$$

and the first formula becomes

$$\cos (b + c) = \cos b \cos c - \sin b \sin c.$$

For $\cos a$, in the second of the original equations, put the value of it from the first: and after an obvious reduction there will result

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A.$$

Introduce now the same hypotheses as before: that is, first let $A = 0$ $\therefore B = 0$ and $a = c - b$

$$\therefore \sin (c - b) = \sin c \cos b - \sin b \cos c;$$

Next let $A = 180° \therefore B = 0 \therefore a = b + c$ and $\cos A = -1$

$$\therefore \sin (c + b) = \sin c \cos b + \sin b \cos c.$$

The two formulas last deduced may be otherwise very readily derived from the preceding pair for $\cos (b \pm c)$. Thus, introducing the proposed hypotheses into the second of the fundamental equations, we have

$$\cos b = \cos (b \pm c) \cos c + \sin (b \pm c) \sin c,$$

that is, replacing $\cos (b \pm c)$ by their values before determined,

$$\cos b = \cos b \cos^2 c \mp \sin b \sin c \cos c + \sin (b \pm c) \sin c,$$
$$\therefore 0 = \cos b (\cos^2 c - 1) \mp \sin b \sin c \cos c + \sin (b \pm c) \sin c,$$

or dividing by $\sin c$, and transposing,

$$\sin (b \pm c) = \sin b \cos c \pm \cos b \sin c.$$

Belfast, Jan. 15, 1844. Y.

SOLUTIONS OF MATHEMATICAL EXERCISES.

I.—*Mr. R. H. Wright, London.*

If a and b be respectively the semi-axes, major and minor, of an ellipse, and a body be projected from one extremity of its axis major, so as to exactly shoot down an inclined plane, meeting the extremity of the axis minor and the other extremity of the axis major; show that, if e be the angle of projection, and $\cos e_1$ the eccentricity of the ellipse,

$$\tan e = 3 \sin e_1.$$

[First Solution.—*Mr. R.E.F. Craufurd, Gent. Cadet, R. M. Academy.*]

Let AB and CD be the major and minor axes of the ellipse, and O its centre; then, A being the point of projection, and CB the plane down which the body is to be projected, it is evident that the line CB must be a tangent to the path of the projectile at the point C. Now since the path of a projectile in free space is a parabola, and since the subtangent is double of the abscissa, it is likewise evident that were the body permitted to pursue its course, instead of descending down the plane CB, it would pass also through I, the point of bisection of OB. Hence the projectile must pass through the two points C and I, whose co-ordinates are respectively (a, b) and $(\frac{3}{2}a, 0)$. Now the equation of the curve described by a projectile in free space is

$$y = x \tan e - \frac{x^2}{4h \cos^2 e} \text{ ....} (1)$$

where h is the altitude due to the velocity of projection, and substituting for x and y the co-ordinates of C and I, we have the two equations

$$b = a \tan e - \frac{a^2}{4h \cos^2 e} \text{ ....} (2)$$

$$0 = \tfrac{3}{2} a \tan e - \frac{\frac{9}{4} a^2}{4h \cos^2 e} \text{ ....} (3)$$

Dividing (2) by a^2, and (3) by $\frac{9}{4}a^2$, and subtracting the latter result from the former, we have at once

$$\tan e = \frac{3b}{a} \text{ ....} (4)$$

Now by the question we have

$$\cos^2 e_1 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} \therefore \sin^2 e_1 = \frac{b^2}{a^2}, \text{ or } \sin e_1 = \frac{b}{a};$$

consequently by (4) we get finally

$$\tan e = 3 \sin e_1.$$

[Second Solution.—*Mr. James Anderson, Montrose.*]

Let v be the velocity of projection, the extremity of the major axis the origin, the major axis the axis of x, and x and y the co-ordinates of the body at any time t; then

$$x = tv \cos e, \text{ and } y = tv \sin e - \tfrac{1}{2} g t^2.$$

Eliminating t, we get for the trajectory of the body, the equation

$$y = x \tan e - \frac{gx^2}{2v^2 \cos^2 e}.$$

But this trajectory is subject to pass through the point a, b; whence

$$b = a \tan e - \frac{ga^2}{2v^2 \cos^2 e},$$

and hence eliminating v,

$$y = x \tan e + x^2 . \frac{b - a \tan e}{a^2}.$$

At the point a, b the inclination of the curve to the axis of x is to be the same as that of the plane, and the tangent of the inclination of the plane is evidently $-\frac{b}{a}$. The inclination of the curve at any point x, y is

$$\frac{dy}{dx} = \tan e + 2x . \frac{b - a \tan e}{a^2},$$

and therefore at the point a, b it is

$$\tan e + \frac{2b}{a} - 2 \tan e, \text{ or } - \tan e + \frac{2b}{a}.$$

Hence $$- \tan e + \frac{2b}{a} = - \frac{b}{a}, \text{ and } \tan e = 3\frac{b}{a}.$$

Now by the definition of eccentricity $\cos e_1 = \frac{\sqrt{a^2 - b^2}}{a}$;

hence $$\sin e_1 = \sqrt{1 - \cos^2 e_1} = \frac{b}{a},$$

and $$\therefore \tan e = 3 \sin e_1.$$

[THIRD SOLUTION.—*Mr. Thomas Weddle, Newcastle-on-Tyne.*]

Take A the point of projection, one extremity of the major axis A B for the origin of horizontal and vertical axes, and let C be the extremity of the minor axis. Then, since $b = a \sin e_1$, B and C are represented by $(2a, 0)$ and $(a, a.\sin e_1)$. The equation of BC is therefore

$$y = - \sin e_1 (x - 2a) \ldots\ldots\ldots\ldots (1)$$

The equations of motion are

$$\frac{d^2y}{dt^2} = -g \ldots. (2) \qquad \frac{d^2x}{dt^2} = 0 \ldots\ldots\ldots\ldots (3)$$

Integrate (2) and (3) successively

$$\therefore \frac{dy}{dt} = u.\sin e - gt \ldots. (4) \qquad \frac{dx}{dt} = u \cos e \ldots\ldots\ldots (5)$$

$$y = u \sin e.t - \tfrac{1}{2}gt^2 \ldots (6) \qquad x = u \cos e.t \ldots\ldots\ldots (7)$$

Here (6) and (7) require no constants, for x, y, and t vanish simultaneously, and those in (4) and (5) are what $\frac{dy}{dt}$ and $\frac{dx}{dt}$ become when $t = 0$, viz. $u.\sin e$ and $u.\cos e$, u being the initial velocity.

From (6) and (7) $$y = \tan e.x - \frac{gx^2}{2u^2\cos^2 e} \dots\dots(8)$$

$$\therefore \frac{dy}{dx} = \tan e - \frac{gx}{u^2\cos^2 e} \dots\dots(9)$$

Now as the body shoots exactly down B C, C is a point in the path of the projectile, and B C a tangent to it. Make, therefore, $x = a$, in (9), and equate the result to the coefficient of x in (1),

$$\therefore \tan e - \frac{ga}{u^2\cos^2 e} = -\sin e_1 \dots\dots(10)$$

Also from (8), C or $(a, a\sin e_1)$ being a point in the curve

$$\tan e.a - \frac{ga^2}{2u^2\cos^2 e} = a.\sin e_1 \dots\dots(11)$$

Divide (11) by $\frac{1}{2}a$, and deduct (10) from the result

$$\therefore \tan e = 3\sin e_1 \dots\dots(12)$$

If a body be projected from A, so as to pass through B $(x_1\, y_1)$ and shoot exactly down B C whose equation is

$$y - y_1 = m(x - x_1) \dots\dots(13)$$

Then instead of (1) we must use (13), and instead of (10), (11), and (12) we shall have

$$\tan e - \frac{gx_1}{u^2\cos^2 e} = m \dots\dots(14)$$

$$\tan e.x_1 - \frac{gx_1^2}{2u^2\cos^2 e} = y_1 \dots\dots(15)$$

and, $$\tan e = \frac{2y_1}{x_1} - m \dots\dots(16)$$

III.—*Mr. Fenwick.*

If the sides and angles of the triangle formed by joining the centres of the escribed circles of a given triangle be denoted by a_1, b_1, c_1, and A_1, B_1, C_1 respectively; and its mass by M_1: then the moment of inertia of the original triangle, with respect to a line perpendicular to its plane through its centre of gravity, is

$$\frac{1}{18} M_1 \cos A_1 \cos B_1 \cos C_1 (a_1^2 \cos^2 A_1 + b_1^2 \cos^2 B_1 + c_1^2 \cos^2 C_1).$$

[First Solution.—*Mr. James Anderson.*]

Bisect the exterior angles of the original triangle ABC by the straight lines AC_1, BC_1, BA_1, CA_1, CB_1, AB_1; then it is easy to shew that A_1, B_1, C_1 are the centres of the escribed circles, and that C_1BA_1, A_1CB_1, and B_1AC_1,

are straight lines. Consequently, $A_1B_1C_1$ is the triangle whose mass is M_1. Also from the construction

$$A_1 = \frac{B+C}{2}; \quad B_1 = \frac{A+C}{2}; \quad C_1 = \frac{A+B}{2};$$

from which we find

$$A = \pi - 2A_1; \quad B = \pi - 2B_1; \quad C = \pi - 2C_1.$$

Also $a_1 = B_1A + AC_1 = b\,\frac{\sin C_1}{\sin B_1} + c\,\frac{\sin B_1}{\sin C_1} = b.\frac{c_1}{b_1} + c.\frac{b_1}{c_1}.$

In a similar way we find

$$b_1 = a.\frac{c_1}{a_1} + c.\frac{a_1}{c_1}, \text{ and } c_1 = a.\frac{b_1}{a_1} + b.\frac{a_1}{b_1}.$$

Deducing the value of a from these three equations, in terms of a_1, b_1, c_1, we find

$$a = a_1.\frac{b^2_1 + c^2_1 - a^2_1}{2b_1c_1} = a_1 \cos A_1.$$

And similarly

$$b = b_1 \cos B_1, \quad \text{and } c = c_1 \cos C_1.$$

Now Euler has shewn that the moment of inertia of a triangle ABC round its centre of gravity is $\frac{M}{36}(a^2 + b^2 + c^2)$, M being its mass. Also (the mass of each square unit being 1),

$$M = \frac{ab}{2} \sin C = \frac{a_1b_1}{2} \cos A_1 \cos B_1 \sin(\pi - 2C_1)$$

$$= \frac{a_1b_1 \sin C_1}{2}. \; 2 \cos A_1 \cos B_1 \cos C_1$$

$$= 2M_1 \cos A_1 \cos B_1 \cos C_1.$$

Hence, by substitution, we have

$$\text{moment of inertia} = \frac{M_1}{18} \cos A_1 \cos B_1 \cos C_1 (a^2_1 \cos^2 A_1 + b^2_1 \cos^2 B_1 + c^2_1 \cos^2 C_1).$$

[SECOND SOLUTION.—γ.]

Let ABC be the original triangle, and $A_1B_1C_1$ the escribed one, or that formed by the intersections of the lines which bisect the exterior angles of the triangle ABC, where A_1, B_1, C_1, are opposite to A, B, C. Join CC_1 and AA_1; and since these lines are evidently at right angles to A_1B_1, B_1C_1, we have

$$B_1C = B_1C_1 \cos B_1 = a_1 \cos B_1 \;\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$B_1A = B_1A_1 \cos B_1 = c_1 \cos B_1 \;\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore \; AC^2 = B_1C^2 + B_1A^2 - 2B_1C.\,B_1A \cos B_1$$

$$= (a^2_1 + c^2_1 - 2a_1c_1 \cos B_1) \cos^2 B_1$$

$$\text{But } \; a^2_1 + c^2_1 - 2a_1c_1 \cos B_1 = A_1C^2_1 = b^2_1$$

$$\therefore \; AC^2 = b^2_1 \cos^2 B_1 \;\; \therefore \; AC = b = b_1 \cos B_1 \ldots\ldots\ldots\ldots (3)$$

Similarly $\quad AB = c = c_1 \cos C_1 \ldots\ldots\ldots\ldots (4)$

$$BC = a = a_1 \cos A_1 \ldots\ldots\ldots\ldots (5)$$

Again, $AB_1C = \frac{1}{2}B_1C.\ B_1A \sin B_1$

$= \frac{1}{2}a_1c_1 \cos^2 B_1 \sin B_1$ by (1, 2)

$= A_1B_1C_1 \cos^2 B_1 \ldots\ldots\ldots\ldots\ldots\ldots (6)$

In a similar manner we get

$CA_1B = A_1B_1C_1 \cos^2 A_1 \ldots\ldots\ldots\ldots\ldots\ldots (7)$

$AC_1B = A_1B_1C_1 \cos^2 C_1 \ldots\ldots\ldots\ldots\ldots\ldots (8)$

$\therefore\ ABC = A_1B_1C_1 - (AB_1C + CA_1B + AC_1B)$

$= A_1C_1B_1 (1 - \cos^2 A_1 - \cos^2 B_1 - \cos^2 C_1)$ by (6,7,8)

$= 2A_1B_1C_1 \cos A_1 \cos B_1 \cos C_1$ (*Hind's Trig.* p. 226.)$\ldots (9)$

Now the moment of inertia (m) of ABC (M being its mass) about the specified axis (see *Earnshaw's Dynamics*, p. 173) is

$$m = \frac{M}{36}(a^2 + b^2 + c^2)$$

$$= \frac{M}{36}(a_1^2 \cos^2 A_1 + b_1^2 \cos^2 B_1 + c_1^2 \cos^2 C_1) \quad \text{by } (3, 4, 5) \ldots (10)$$

And since $ABC : A_1B_1C_1 :: M : M_1$

$$\therefore\ M = M_1 \frac{ABC}{A_1B_1C_1} = 2M_1 \cos A_1 \cos B_1 \cos C_1, \text{ by } (9).$$

Hence (10) becomes by substitution

$$m = \frac{1}{18} M_1 \cos A_1 \cos B_1 \cos C_1 (a_1^2 \cos^2 A_1 + b_1^2 \cos^4 B_1 + c_1^2 \cos^2 C_1).$$

Scholium.—The relation (9) between the primitive triangle and the escribed one is not given in the "*Horæ Geom.*"

[THIRD SOLUTION.—*Mr. Weddle.*]

Taking the figure employed by Mr. Davies, in the *Horæ Geom.*, of the Lady's Diary, and adopting his notation, we have from p. 88, *Diary* for 1842, $A_1 = \frac{1}{2}(\pi - A)$, $B_1 = \frac{1}{2}(\pi - B)$, and $C_1 = \frac{1}{2}(\pi - C)$. Moreover the diameter of the circle circumscribing the triangle $O_1O_2O_3$ being 2D, we have

$a_1 = 2D \sin A_1$, but $a = D. \sin A = D. \sin 2A_1 = 2D. \cos A_1. \sin A_1$.

$\therefore\ a = a_1 \cos A_1.$

Similarly $b = b_1 \cos B_1.$

and $c = c_1 \cos C_1.$

Again, M being the mass of the original triangle, we have (*ib.* p. 87)

$$M = M_1. \frac{r}{D} = (\textit{ib. } \text{p. 82, Cor. 5})\ 2M_1 \sin\tfrac{1}{2}A. \sin\tfrac{1}{2}B. \sin\tfrac{1}{2}C$$

$$= 2M_1 \cos A_1 \cos B_1 \cos C_1.$$

Now it is shown in *Earnshaw's Dynamics*, 2nd Edit. p. 173, that the required moment of inertia is

$$\frac{M}{36}\{a^2 + b^2 + c^2\},$$

which by substitution becomes

$$\frac{M_1}{18} \cos A_1 \cos B_1 \cos C_1 \{a_1^2 \cos^2 A_1 + b_1^2 \cos^2 B_1 + c_1^2 \cos^2 C_1\}.$$

IV.—*Mr. Rutherford.*

If α, β are the co-ordinates of the centre of a circle which cuts an ellipse in four points, and the equations of the ellipse and circle be

$$a^2y^2 + b^2x^2 = a^2b^2, \text{ and } (y-\beta)^2 + (x-\alpha)^2 = r^2,$$

then the product of the distances of the four points of intersection from the major axis is

$$\frac{b^4\{(a^2+\alpha^2+\beta^2-r^2)^2 - 4a^2\alpha^2\}}{(a^2-b^2)^2}.$$

[First Solution.—*Mr. Thomas Weddle.*]

The equations of the ellipse and circle are

$$a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$y^2 + x^2 - 2\beta y - 2\alpha x = r^2 - \alpha^2 - \beta^2 \ldots\ldots\ldots\ldots\ldots(2)$$

Multiply (2) by b^2, and deduct the product from (1); then

$$(a^2-b^2)y^2 + 2\beta b^2 y + 2\alpha b^2 x = b^2(a^2+\alpha^2+\beta^2-r^2).$$

Let $a^2 - b^2 = c^2$, and $a^2+\alpha^2+\beta^2-r^2 = m^2$, and transpose; therefore

$$2\alpha b^2 x = b^2m^2 - 2\beta b^2 y - c^2y^2.$$

Square each side, and we have

$$4\alpha^2b^4x^2 = b^4m^4 - 4\beta b^4m^2y + (4\beta^2b^2 - 2c^2m^2)b^2y^2 + 4\beta b^2c^2y^3 + c^4y^4 \ldots(3)$$

Eliminate x^2 from (1) and (3), and reduce, then

$$c^4y^4 + 4\beta b^2c^2y^3 + (4\alpha^2a^2 + 4\beta^2b^2 - 2c^2m^2)b^2y^2 - 4\beta b^4m^2y$$
$$+ b^4(m^4 - 4\alpha^2a^2) = 0 \ldots(4)$$

Hence if y_1, y_2, y_3, and y_4 be the four values of y, we shall have, by the theory of equations,

$$y_1y_2y_3y_4 = \frac{b^4(m^4 - 4\alpha^2a^2)}{c^4} = \frac{b^4\{(a^2+\alpha^2+\beta^2-r^2)^2 - 4\alpha^2a^2\}}{(a^2-b^2)^2}.$$

Again, multiply (2) by a^2, and subtract (1) from the product; then, if $b^2+\alpha^2+\beta^2-r^2 = n^2$, we have, after transposing,

$$2\beta a^2 y = a^2n^2 - 2\alpha a^2 x - c^2x^2.$$

Square each side, and eliminate y^2 from the result by means of eq. (1); then we get

$$c^4x^4 + 4\alpha a^2c^2x^3 + (4\alpha^2a^2 + 4\beta^2b^2 - 2c^2n^2)a^2x^2 - 4\alpha a^4n^2x$$
$$+ a^4(n^4 - 4\beta^2b^2) = 0 \ldots\ldots\ldots(5)$$

And if x_1, x_2, x_3, x_4 are the four values of x, we have

$$x_1x_2x_3x_4 = \frac{a^4(n^4 - 4\beta^2b^2)}{c^4} = \frac{a^4\{(b^2+\alpha^2+\beta^2-r^2)^2 - 4\beta^2b^2\}}{(a^2-b^2)^2};$$

which gives the product of the distances of the four points of intersection from the minor axis; whence

$$\frac{x_1x_2x_3x_4}{y_1y_2y_3y_4} = \frac{a^4}{b^4}\cdot\frac{\{(b^2+\alpha^2+\beta^2-r^2)^2 - 4\beta^2b^2\}}{\{(a^2+\alpha^2+\beta^2-r^2)^2 - 4\alpha^2a^2\}}$$

$$= \frac{a^4}{b^4}\cdot\frac{\{(b+\beta)^2+\alpha^2-r^2\}.\{(b-\beta)^2+\alpha^2-r^2\}}{\{(a+\alpha)^2+\beta^2-r^2\}.\{(a-\alpha)^2+\beta^2-r^2\}}.$$

[SECOND SOLUTION.—*Mr. James Anderson.*]

By the equations of the ellipse and circle, we have

$$x = \frac{a}{b}\sqrt{b^2 - y^2}, \text{ and } x = \sqrt{r^2 - (y-\beta)^2} + \alpha;$$

Equating the squares of these values, gives

$$\frac{a^2}{b^2}(b^2 - y^2) = r^2 - (y-\beta)^2 + \alpha^2 + 2\alpha\sqrt{r^2 - (y-\beta)^2},$$

$$\text{or} \quad a^2(b^2 - y^2) - b^2r^2 + b^2(y-\beta)^2 - \alpha^2b^2 = 2\alpha b^2\sqrt{r^2 - (y-\beta)^2}.$$

Squaring, dividing by $(a^2 - b^2)^2$, and transposing, we find a result of the form

$$y^4 + Ay^3 + By^2 + Cy + D = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where D, the term independent of y, is found after reduction to be

$$\frac{b^4\{(a^2 + \alpha^2 + \beta^2 - r^2)^2 - 4a^2\alpha^2\}}{(a^2 - b^2)^2}.$$

If the biquadratic equation (1) has four real roots, the curves intersect in four points, whose ordinates correspond to these four values of y. Suppose the roots to be real, and to be y_1, y_2, y_3, y_4; then by the theory of equations

$$(y - y_1)(y - y_2)(y - y_3)(y - y_4) = 0$$

is an equation identical with (1). Comparing the terms we find

$$y_1y_2y_3y_4 = D = \frac{b^4\{(a^2 + \alpha^2 + \beta^2 - r^2)^2 - 4a^2\alpha^2\}}{(a^2 - b^2)^2}.$$

In the same manner, by eliminating y from the proposed equations (1) and (2), we find a corresponding expression for the product of the distances of the four points of intersection from the minor axis.

V.—*Mr. Fenwick.*

From the angles A, B, C, of a triangle, draw lines through any point P, to meet the opposite sides in D, E, F; and join DE, EF, FD, meeting CF, AD, BE, in f, d, e. Having joined fd, de, ef, draw lines from A, B, C, and D, E, F, to bisect EF, DF, DE, and ef, df, de, respectively; the former meeting in M, and the latter in N; then will M, N, P range in a straight line.

[FIRST SOLUTION.—*Mr. Weddle.*]

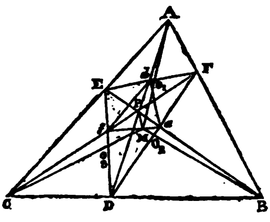
Let AP $= m$, DP $= m_1$, BP $= n$, and PE $= n_1$. Take BP, AP for axes.

The equation of AB is $\frac{x}{n} + \frac{y}{m} = 1 \ldots (1)$

$\ldots\ldots\ldots\ldots$ AC .. $-\frac{x}{n_1} + \frac{y}{m} = 1 \ldots (2)$

$\ldots\ldots\ldots\ldots$ BC .. $\frac{x}{n} - \frac{y}{m_1} = 1 \ldots (3)$

$-\frac{x}{n_1} - \frac{y}{m_1} = 1 \ldots (4)$



Deduct (2) from (3), hence

equation of CF is, $\left(\frac{1}{n}+\frac{1}{n_1}\right)x-\left(\frac{1}{m}+\frac{1}{m_1}\right)y=0 \ldots\ldots (5)$

Multiply (5) by λ, and add the result to (1), then

$$\left(\frac{1}{n}+\frac{\lambda}{n}+\frac{\lambda}{n_1}\right)x+\left(\frac{1}{m}-\frac{\lambda}{m}-\frac{\lambda}{m_1}\right)y=1;$$

This is the equation of any line through F, if it is to be restricted to EF it must pass through E or $(-n_1, 0)$, which requires that $\lambda=-1$; hence

Equation of EF is $-\frac{x}{n_1}+\left(\frac{2}{m}+\frac{1}{m_1}\right)y=1 \ldots\ldots\ldots\ldots (6)$

Similarly .... DF .. $\left(\frac{2}{n}+\frac{1}{n_1}\right)x-\frac{y}{m_1}=1 \ldots\ldots\ldots\ldots (7)$

From (1) and (5) the co-ordinates of F are

$$x'=\frac{\frac{1}{m}+\frac{1}{m_1}}{\frac{2}{nm}+\frac{1}{nm_1}+\frac{1}{n_1m}},\ y'=\frac{\frac{1}{n}+\frac{1}{n_1}}{\frac{2}{nm}+\frac{1}{nm_1}+\frac{1}{n_1m}} \ldots (8)$$

Hence denoting the middle points O_1, O_2, O_3 of EF, DF and DE by (x_1, y_1), (x_2, y_2) and (x_3, y_3) we have

$$x_1=\tfrac{1}{2}(x'-n_1)=\frac{n_1\left(\frac{1}{n_1m_1}-\frac{2}{nm}-\frac{1}{nm_1}\right)}{2\left(\frac{2}{nm}+\frac{1}{nm_1}+\frac{1}{n_1m}\right)};\ y_1=\tfrac{1}{2}y'=\frac{\frac{1}{n}+\frac{1}{n_1}}{2\left(\frac{2}{nm}+\frac{1}{nm_1}+\frac{1}{n_1m}\right)}$$

$$x_2=\tfrac{1}{2}x'=\frac{\frac{1}{m}+\frac{1}{m_1}}{2\left(\frac{2}{nm}+\frac{1}{nm_1}+\frac{1}{n_1m}\right)};\ y_2=\tfrac{1}{2}(y'-m_1)=\frac{m_1\left(\frac{1}{n_1m_1}-\frac{2}{nm}-\frac{1}{n_1m}\right)}{2\left(\frac{2}{nm}+\frac{1}{nm_1}+\frac{1}{n_1m}\right)}$$

$x_3=-\tfrac{1}{2}n_1, \quad y_3=-\tfrac{1}{2}m_1.$

Also if (α, β) denote the point C, we have from (2) and (3)

$$\alpha=\frac{\frac{1}{m}+\frac{1}{m_1}}{\frac{1}{nm}-\frac{1}{n_1m_1}},\quad \beta=\frac{\frac{1}{n}+\frac{1}{n_1}}{\frac{1}{nm}-\frac{1}{n_1m_1}}$$

The equations of AO_1, BO_2, and CO_3, which pass through the points (x_1, y_1) and (o, m), (x_2, y_2) and (n, o), and (x_3, y_3) and (α, β) respectively, will be found after reduction to be as follows, viz.

$$-\left(\frac{3}{nm}+\frac{2}{nm_1}+\frac{1}{n_1m}\right)\frac{x}{n_1}+\left(\frac{2}{nm}+\frac{1}{nm_1}-\frac{1}{n_1m_1}\right)\frac{y}{m}=\frac{2}{nm}+\frac{1}{nm_1}-\frac{1}{n_1m_1} \ldots (9)$$

$$\left(\frac{2}{nm}+\frac{1}{n_1m}-\frac{1}{n_1m_1}\right)\frac{x}{n}-\left(\frac{3}{nm}+\frac{2}{n_1m}+\frac{1}{nm_1}\right)\frac{y}{m_1}=\frac{2}{nm}+\frac{1}{n_1m}-\frac{1}{n_1m_1} \ldots (10)$$

$$\left(\frac{1}{nm}+\frac{2}{nm_1}+\frac{1}{n_1m_1}\right)\frac{x}{n_1}-\left(\frac{1}{nm}+\frac{2}{n_1m}+\frac{1}{n_1m_1}\right)\frac{y}{m_1}=\frac{1}{n_1m}-\frac{1}{nm_1} \ldots\ldots (11)$$

Deduct (11) from the sum of (10) and $\frac{m}{m_1}$ times (9)

$$\therefore\ x = Q\left(\frac{1}{m}+\frac{1}{m_1}\right)\left(\frac{2}{nm}+\frac{1}{nm_1}-\frac{1}{n_1m_1}\right)\ldots\ldots\ (12)$$

where $\frac{1}{2Q}=\frac{1}{n^2m^2}-\frac{1}{nn_1m_1^2}-\frac{1}{n_1^2mm_1}-\frac{3}{nn_1mm_1}$.

Similarly $y=Q\left(\frac{1}{n}+\frac{1}{n_1}\right)\left(\frac{2}{nm}+\frac{1}{n_1m}-\frac{1}{n_1m_1}\right)\ldots\ldots\ldots\ldots\ (13)$

Also, taking the sum of (9) and (11) from (10), we have

$$\left(\frac{1}{n}+\frac{1}{n_1}\right)\left(\frac{2}{nm}+\frac{1}{n_1m}-\frac{1}{n_1m_1}\right)x=\left(\frac{1}{m}+\frac{1}{m_1}\right)\left(\frac{2}{nm}+\frac{1}{nm_1}-\frac{1}{n_1m_1}\right)y\ldots(14)$$

Now, (12) and (13) satisfy (14), hence AO_1, BO_2, CO_3 intersect in some point M, and (14) is therefore the equation of PM. It may be shown in precisely the same manner that the other lines mentioned meet in some point N. Let $x=0$ in (6), and $y=0$ in (7);

$$\therefore\ \frac{1}{Pd}=\frac{2}{m}+\frac{1}{m_1},\ \text{and}\ \frac{1}{Pe}=\frac{2}{n}+\frac{1}{n_1}\ldots\ldots\ldots\ldots\ (15)$$

Now we shall evidently get the equation of PN, if in (14), we substitute $-m_1$ for m, $-n_1$ for n, Pd for $-m_1$ and Pe for $-n_1$. This will be found, after reduction, to re-produce (14): hence PM, PN coincide, or P, M, N range in a straight line.

From the preceding equations several properties may be deduced. It appears from (15) that AD is harmonically divided in P and d, and BE in P and e. In a similar manner it may be shown that CP is harmonically divided in P and f. Hence

(A) "If lines be drawn from the angular points of a triangle through any point to meet the opposite sides (produced if necessary) and the points of section be joined, the former lines will be cut harmonically."

Let BC and EF produced meet in P, AC and DF in Q, and AB and DE in R. By (3) and (6) the equation of any line through P is

$$\left(\frac{\lambda}{n}-\frac{1}{n_1}\right)x+\left(-\frac{\lambda}{m_1}+\frac{2}{m}+\frac{1}{m_1}\right)y=1+\lambda,$$

and by (1) and (4) the equation of any line through R is

$$\left(\frac{\lambda_1}{n}-\frac{1}{n_1}\right)x+\left(\frac{\lambda_1}{m}-\frac{1}{m_1}\right)y=1+\lambda_1.$$

Now if both these represent PR we must have

$$\frac{\frac{\lambda}{n}-\frac{1}{n_1}}{\frac{\lambda_1}{n}-\frac{1}{n_1}}=\frac{-\frac{\lambda}{m_1}+\frac{2}{m}+\frac{1}{m_1}}{\frac{\lambda_1}{m}-\frac{1}{m_1}}=\frac{1+\lambda}{1+\lambda_1}$$

$\therefore\ \lambda=\lambda_1=2$; hence the equation of PR is

$$\left(\frac{2}{n}-\frac{1}{n_1}\right)x+\left(\frac{2}{m}-\frac{1}{m_1}\right)y=3\ \ldots\ldots\ldots\ldots\ldots\ (16)$$

By a similar process we shall find the equation of QR to be the same. Hence

(B). "If through any point, lines be drawn from the angular points of a triangle to meet the opposite sides, and if lines, joining the points of section, be produced to meet the sides produced, the three points of concourse range in a straight line."

Again by (15), the equation of de is

$$\left(\frac{2}{n}+\frac{1}{n_1}\right)x+\left(\frac{2}{m}+\frac{1}{m_1}\right)y=1\dots\dots\dots\dots\dots(17)$$

Now if we deduct (4) from twice (1) we shall get (17). Hence AB, DE, de meet in R; similarly AC, DF, df meet in Q, and BC, EF, ef in P. Hence, by generalizing, we readily get the following theorem.

(C). "From the angular points A, B, C of a triangle draw through any point O, the lines AO, BO, CO, to meet the opposite sides in A_1, B_1, C_1 respectively. Let B_1C_1, A_1C_1, A_1B_1 be joined, intersecting the three lines AO, BO, CO, in A_2, B_2, C_2; let B_2C_2, A_2C_2, A_2B_2 be joined, intersecting the same three lines in A_3, B_3, C_3; and so on. Then shall BC, B_1C_1, B_2C_2, B_3C_3, etc., intersect in some point P; AC, A_1C_1, A_2C_2, A_3C_3 etc., in some point Q, and AB, A_1B_1, A_2B_2, A_3B_3, etc., in some point R: and the three points of intersection P, Q, R shall range in a straight line."

Propositions (A) and (B) are well known theorems: (C) was, so far as I know, first proposed by me in the last number of the Northumbrian Mirror, and was therefore unanswered. I have likewise discovered two analogous theorems which I shall add.

(D) Proposition (C) is true of spherical triangles, substituting the arcs of great circles for straight lines.

At the point O let a tangent plane (to the sphere) be drawn, from the centre X, draw XA, XB, XC, XA_1, *etc.*, meeting the tangent plane in A', B', C', A'_1, *etc.* By (C) the straight lines $B'C'$, $B'_1C'_1$, $B'_2C'_2$, *etc.*, meet in some point P', $A'C'$, $A'_1C'_1$, $A'_2C'_2$, *etc.* in Q', and $A'B'$, $A'_1B'_1$, $A'_2B'_2$, *etc.* in R'. Let XP', XQ', XR' meet the sphere in P, Q, R. Since $B'C'$, $B'_1C'_1$, $B'_2C'_2$, *etc.*, meet in P', the planes $XB'C'$, $XB'_1C'_1$, $XB'_2C'_2$, *etc.*, intersect in the line XPP', hence BC, B_1C_1, B_2C_2, *etc.*, which are arcs of great circles in these planes must meet the line XPP', where it cuts the sphere, that is in P.

In a similar way the other two sets of arcs are proved to meet in Q and R. Also since the points P', Q', R' are in a straight line, the lines XPP', XQQ', XRR' are in a plane which passes through the centre, that is, P, Q, R are in a great circle of the sphere.

(E.) From A, B, C and D the angular points of a triangular pyramid, let lines AO, BO, CO and DO, be drawn through any point O to meet the opposite faces in A_1, B_1, C_1, D_1 respectively; take these points for the angles of another pyramid, and let its faces meet the four preceding lines in A_2, B_2, C_2, D_2; take these points for the angles of another pyramid, and let its faces meet the same lines in A_3, B_3, C_3 and D_3, and so on. Then will CD, C_1D_1, C_2D_2, *etc.* converge to some point P; BD, B_1D_1, B_2D_2, *etc.* to Q; AD, A_1D_1, A_2D_2, *etc.* to R; BC, B_1C_1, B_2C_2, *etc.* to S; AC, A_1C_1, A_2C_2, *etc.*, to T; and AB, A_1B_1, A_2B_2, *etc.* to U. Also the six points of intersection are in the same plane.

Since the lines AA_1, BB_1 pass through O, they are in the same plane, hence AB, A_1B_1 will intersect in U. In like manner AC, A_1C_1 intersect in T and

BC, B_1C_1 in S; also S,T,U are in a straight line, being in the intersection of the planes ABC, and $A_1B_1C_1$. Similarly A_1B_1, A_2B_2 will intersect in U', A_1C_1, A_2C_2, in T' and B_1C_1, B_2C_2 in S', and the straight line $S'T'U'$ will be the intersection of the planes $A_1B_1C_1$, $A_2B_2C_2$. Let the plane $A_1B_1C_1$ intersect DA, DB, and DC in a, b, and c; then, since AB and ab are in the same plane (DAB) and the former in ABC and the latter in $A_1B_1C_1$; AB, ab and STU will intersect in the same point, but AB, A_1B_1 and STU intersect in U, hence ab and A_1B_1 intersect in U. Again, join D_1A_2, D_1B_2, D_1C_2 meeting B_1C_1, A_1C_1 and A_1B_1 in a', b', c', hence A_2B_2 and $a'b'$ being in the same plane $(A_2D_1B_2)$ and the former in $A_2B_2C_2$ and the latter in $A_1B_1C_1$, A_2B_2, $a'b'$ and $S'T'U'$ intersect in the same point, but A_1B_1, A_2B_2 and $S'T'U'$ intersect in U'; hence A_1B_1 and $a'b'$ intersect in U'. Let DO cut $A_1B_1C_1$ in O'. Since the points A, A_1, A_2, O; D, D_1, O,O'; D, A, a, and D_1, A_2, a' are respectively in straight lines, a moment's consideration will show that they are all in the same plane, but the points a, A_1, a', O' are also in the plane $A_1B_1C_1$, they are therefore in the same straight line. Similarly, b, B_1, b', O' are in a straight line, as also c, C_1, c', O'. Moreover each of the triads of points a, b, C_1; a, c, B_1; b, c, A_1; B_1, C_1, a'; A_1, C_1, b'; and A_1, B_1, c', is in a straight line; hence abc, $A_1B_1C_1$, $a'b'c'$ are three such triangles as are mentioned in (C). Hence ab, A_1B_1, $a'b'$ intersect in the same point, but the two first intersect in U, and the two last in U'; hence U, U' coincide, consequently AB, A_1B_1, A_2B_2 intersect in the same point U. Now if we consider A_1, B_1, C_1, D_1 to be the primitive pyramid, we can, in like manner, show that A_1B_1, A_2B_2, A_3B_3 meet in the same point, and so on. Hence AB, A_1B_1, A_2B_2, A_3B_3, &c., meet in the same point U. In the same manner the other edges may be proved to intersect, as stated in the proposition. Again, it was shown above, that S, T, and U are in a straight line, in the same manner it may be demonstrated that Q, R, U; P, R, T; and P, Q, S, are respectively in straight lines, the last three lines by their intersection form the triangle PQR, and they are therefore in the same plane. Hence the points P, Q, R, S, T, U are in the same plane.

The proposer of this theorem (Exercise *v.*) deduces also the following :—

(F) From the angles of A,B,C of a triangle draw lines through any point P to meet the opposite sides in A_1,B_1,C_1. Let B_1C_1, A_1C_1, A_1B_1 be joined, intersecting the lines AP, BP, CP in A_2, B_2, C_2; let B_2C_2, A_2C_2, A_2B_2 be joined, intersecting the same three lines in A_3,B_3,C_3; and so on. Then if the sides of the triangle ABC be tangents to a conic section in A_1,B_1,C_1; the sides of the triangle $A_1B_1C_1$ be tangents to a conic section in A_2,B_2,C_2; and so on; the centres of all these conic sections will be in a straight line passing through P.

This remarkable property follows at once from the theorem in question, and the two following well known properties of the conic sections:

(1) "That the three straight lines drawn from the vertices of a triangle described about a conic section to the points of contact opposite intersect in a point."

(2) "When two tangents to a conic section are drawn, the point of intersection of these tangents, the middle point of the chord of contact and the centre of the conic section, are in the same straight line."

[Second Solution.—*Mr. John Laws, Newcastle-on-Tyne.*]

Take PD, PC, as axes of x and y; and denote the several points thus:

(D)............	$a_1, 0$	(C)...........	$0, b_2$
(d)............	$-a_2, 0$	(F)............	$0, -b_3$
(A)............	$-a_3, 0$	(E)............	$-x_1, y_1$
(f)...........	$0, b_1$	(e)............	$x_2, -y_2$

Then the subsequent equations readily follow:

(AM).......... $y(2a_3 - x_1) = (y_1 - b_3)(x + a_3)$ .... (1)

(DN).......... $y(x_2 - 2a_1) = (b_1 - y_2)(x - a_1)$ .... (2)

(CM).... $(y - b_2)(a_1 - x_1) = (y_1 - 2b_2)x$ .... (3)

(FN).... $(y + b_3)(x_2 - a_2) = (2b_3 - y_2)x$ .... (4)

(CD).. $\dfrac{y}{b_2} + \dfrac{x}{a_1} = 1$....(5) | (D$f$).... $\dfrac{y}{b_1} + \dfrac{x}{a_1} = 1$ ....(7)

(AF)..$-\dfrac{y}{b_3} - \dfrac{x}{a_3} = 1$....(6) | (F$d$) .. $-\dfrac{y}{b_3} - \dfrac{x}{a_2} = 1$ ....(8)

Also, since the following triads of points are situated in a straight line, viz. the points A, E, C; D, e, F; E, d, F; E, f, D; E, P, e; we have the following relations:

$a_3y_1 + b_2x_1 = a_3b_2$ ....(9)

$a_1y_2 + b_3x_2 = a_1b_3$ ....(10)

$b_3x_1 - a_2y_1 = a_2b_3$ ....(11)

$a_1y_1 - b_1x_1 = a_1b_1$ ....(12)

$x_1y_2 = x_2y_1$ ....(13)

Combining (1, 3), and also (2, 4), so that the absolute terms in the resulting equations may be eliminated, we obtain the equations of PM, PN, keeping in mind the relations (9, 10), viz.

$$\text{(PM)}..y\left\{\frac{a_3(b_2+b_3)}{y_1-b_3}\right\} + x\left\{\frac{b_2(a_1+a_3)}{x_1-a_1}\right\} = 0 \text{............(14)}$$

$$\text{(PN)}..y\left\{\frac{a_1(b_1+b_3)}{b_1-y_2}\right\} + x\left\{\frac{b_3(a_1+a_2)}{a_2-x_2}\right\} = 0 \text{............(15)}$$

Now it remains to be proved that

$$\alpha\beta_1 = \alpha_1\beta \text{.................... ....(16)}$$

where $\alpha\beta$; $\alpha_1\beta_1$, are the coefficients of y and x in (14) and (15) respectively.

Taking (6) from (5), and also (8) from (7), we get the equations of PB and PE, viz.

$$\text{(PB)}....y\left\{\frac{1}{b_2} + \frac{1}{b_3}\right\} + x\left\{\frac{1}{a_1} + \frac{1}{a_3}\right\} = 0,$$

$$\text{(PE)}....y\left\{\frac{1}{b_1} + \frac{1}{b_3}\right\} + x\left\{\frac{1}{a_1} + \frac{1}{a_2}\right\} = 0;$$

but the points P, B, E are in the same straight line

$$\therefore \left(\frac{1}{a_1} + \frac{1}{a_2}\right)\left(\frac{1}{b_2} + \frac{1}{b_3}\right) = \left(\frac{1}{a_1} + \frac{1}{a_3}\right)\left(\frac{1}{b_1} + \frac{1}{b_3}\right),$$

or, $$a_3(a_1 + a_2)(b_2 + b_3) = \frac{a_2b_2}{b_1}(a_1 + a_3)(b_1 + b_3) \text{..............(17)}$$

Again, from (10, 13) we get

$$x_2 = \frac{a_1 b_3 x_1}{a_2 y_1 + b_3 x_1}, \text{ and } y_2 = \frac{a_2 b_1 y_1}{a_2 y_1 + b_3 x_1} \dots\dots\dots\dots\dots\dots(18)$$

$\therefore$ (18, 12)

$$b_1 - y_2 = \frac{a_2 b_3 y_1 - b_1(a_2 y_1 - b_3 x_1)}{a_2 y_1 + b_3 x_1} = \frac{a_2 b_1(y_1 - b_3)}{a_2 y_1 + b_3 x_1} \dots\dots\dots\dots(19)$$

Similarly from (18, 11) we get

$$a_2 - x_2 = \frac{a_2 b_3(x_1 - a_1)}{a_2 y_1 + b_3 x_1} \dots\dots\dots\dots\dots\dots(20)$$

Eliminate $a_2 y_1 + b_3 x_1$ from (19, 20), and we obtain

$$(y_1 - b_3)(a_2 - x_2) = \frac{a_2 b_3}{a_2 b_1}(b_1 - y_2)(x_1 - a_1) \dots\dots\dots\dots\dots(21)$$

Hence using the expressions for α, β_1, and keeping in mind (17, 21), we have

$$\alpha\beta_1 = \frac{a_2 b_2(a_1 + a_2)(b_2 + b_3)}{(y_1 - b_3)(a_2 - x_2)} = \frac{a_1 b_2(a_1 + a_3)(b_1 + b_3)}{(b_1 - y_2)(x_1 - a_1)} \dots\dots(22)$$

But
$$\alpha_1\beta = \frac{a_1 b_2(a_1 + a_3)(b_1 + b_3)}{(b_1 - y_2)(x_1 - a_1)} \dots\dots\dots\dots\dots\dots(23)$$

Hence, since (22) and (23) are equal, M, N, P range in the same straight line.

VII.—*Mr. Rutherford.*

Find the magnitude and position of a circle which touches three given circles in mutual contact described on the surface of a given sphere.

[First Solution.—*Mr. James Anderson.*]

If any one of the given circles lies on the smaller of the two segments into which the sphere is divided by either of the other two, the problem becomes unlimited, and does not require consideration. If not, let A, B, and C be their *nearest* poles, and r_1, r_2, r_3 their respective polar distances. It is evident that A, B, C will all lie on one of the segments of the sphere cut off by the circle sought. To fix the thoughts, let O be its pole, which lies in the other segment, and let r_4 be its polar distance: then, in every case,

$$AB = r_1 + r_2;\quad BC = r_2 + r_3;\quad AC = r_1 + r_3;$$
$$AO = r_1 + r_4;\quad BO = r_2 + r_4;\quad CO = r_3 + r_4.$$

Let the angle BAO be designated by θ_1 and CAO by θ_2; θ_1 being positive when measured from AB towards AC, and θ_2 positive when measured from AC to AB; then in every case $\theta_2 = A - \theta_1$, and consequently

$$\sin^2\frac{\theta_2}{2} = \sin^2\frac{A - \theta_1}{2} = \sin^2\frac{A}{2}\cos^2\frac{\theta_1}{2} + \cos^2\frac{A}{2}\sin^2\frac{\theta_1}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}.$$

Transposing so as to have $2\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}$ standing alone on one side, squaring both sides, and transposing all the terms to one side, we have, after very obvious reductions,

$$\left(\sin^2\frac{A}{2} - \sin^2\frac{\theta_1}{2}\right)^2 - \sin^2\frac{\theta_2}{2}\left\{2\left(\sin^2\frac{A}{2}\cos^2\frac{\theta_1}{2} + \cos^2\frac{A}{2}\sin^2\frac{\theta_1}{2}\right) - \sin^2\frac{\theta_2}{2}\right\} = 0.$$

Giving to the functions of $\frac{A}{2}, \frac{\theta_1}{2}, \frac{\theta_2}{2}$, their values in terms of the sides of spherical triangles ABC, AOB, CAO, viz.

$$\sin^2\frac{A}{2} = \frac{\sin r_2 \sin r_3}{\sin(r_1+r_2)\sin(r_1+r_3)};\ \cos^2\frac{A}{2} = \frac{\sin r_1 \sin(r_1+r_2+r_3)}{\sin(r_1+r_2)\sin(r_1+r_3)};$$

$$\sin^2\frac{\theta_1}{2} = \frac{\sin r_2 \sin r_4}{\sin(r_1+r_2)\sin(r_1+r_4)};\ \cos^2\frac{\theta_1}{2} = \frac{\sin r_1 \sin(r_1+r_2+r_4)}{\sin(r_1+r_2)\sin(r_1+r_4)};$$

$$\text{and,}\quad \sin^2\frac{\theta_2}{2} = \frac{\sin r_3 \sin r_4}{\sin(r_1+r_3)\sin(r_1+r_4)};$$

and multiplying by $\sin^2(r_1+r_2)\sin^2(r_1+r_3)\sin^2(r_1+r_4)$, we have

$$\sin^2 r_2\{\sin r_3 \sin(r_1+r_4) - \sin r_4 \sin(r_1+r_3)\}^2$$
$$-2\sin r_1 \sin r_2 \sin r_3 \sin r_4\{\sin r_3 \sin(r_1+r_2+r_4) + \sin r_4 \sin(r_1+r_2+r_3)\}$$
$$+\sin^2 r_3 \sin^2 r_4 \sin^2(r_1+r_2) = 0.$$

Write $\sin r_1 \sin(r_3-r_4)$ for $\sin r_3 \sin(r_1+r_4) - \sin r_4 \sin(r_1+r_3)$; develope the series of the compound arcs, and divide by $\sin^2 r_1 \sin^2 r_2 \sin^2 r_3 \sin^2 r_4$, then there results without further reduction

$$\cot^2 r_3 + \cot^2 r_4 - 2\cot r_3 \cot r_4 - 2(2\cot r_1 \cot r_2 + \cot r_1 \cot r_3 + \cot r_2 \cot r_3$$
$$+ \cot r_1 \cot r_4 + \cot r_2 \cot r_4 - 2) + \cot^2 r_1 + \cot^2 r_2 + 2\cot r_1 \cot r_2 = 0.$$

Consequently,

$$\cot^2 r_4 - 2\cot r_4(\cot r_1 + \cot r_2 + \cot r_3) = 2(\cot r_1 \cot r_2 + \cot r_1 \cot r_3 + \cot r_2 \cot r_3 - 2)$$
$$-(\cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3),$$

and

$$\cot r_4 = \cot r_1 + \cot r_2 + \cot r_3 \pm 2\sqrt{(\cot r_1 \cot r_2 + \cot r_1 \cot r_3 + \cot r_2 \cot r_3 - 1)}\ \ldots\ (1)$$

The surd part of this result cannot become unreal unless $r_1+r_2+r_3$ be greater than π, which is impossible, since $r_1+r_2+r_3$ is the semiperimeter of a spherical triangle; hence there are two tangent circles. Let r_4 and r_5 be their respective polar distances; then

$$\cot r_4 + \cot r_5 = 2(\cot r_1 + \cot r_2 + \cot r_3).$$

If $r_1+r_2+r_3 = \pi$, the surd part of (1) vanishes, and r_4 and r_5 are then equal. This is the case when the poles of the three given circles are in a great circle of the sphere.

Denoting the rational part of (1) by m, and the surd part by n, we have

$$\cot r_4 = m + n,\ \text{and}\ \cot r_5 = m - n.$$

Hence r_4 and r_5 are each less than $\frac{\pi}{2}$ when m is greater than n, and r_5 is greater than $\frac{\pi}{2}$ when m is less than n. In the latter case r_5 is the distance of the fifth circle from its most remote pole, whilst r_4 is in every case the distance of the fourth circle from its nearest pole. When r_5 is greater than $\frac{\pi}{2}$, let ρ_5 denote the distance of the fifth circle from its nearest pole, and then

$$\cot r_4 - \cot \rho_5 = 2(\cot r_1 + \cot r_2 + \cot r_3).$$

This occurs when the least of the three given circles cuts either of the great circles, which may be drawn touching the other two.

If r_1, r_2, r_3 be infinitely small compared to the radius of the sphere, we

shall have the case of three tangent circles on a plane. We must then evidently write $\frac{1}{r_1}$ for cot r_1, $\frac{1}{r_2}$ for cot r_2, *etc.*, and there will result

$$\frac{1}{r_4}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\sqrt{\left(\frac{1}{r_1r_2}+\frac{1}{r_1r_3}+\frac{1}{r_2r_3}\right)};$$

also, $\frac{1}{r_4}+\frac{1}{r_5}=2\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$,

when the smallest of the three given circles does not cut the straight line touching the other two. If it does cut the tangent line, we have, as in the case of the three circles on a spherical surface,

$$\frac{1}{r_4}-\frac{1}{\rho_5}=2\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right),$$

a property already enunciated in the former number of the Mathematician.

It is almost unnecessary to add that, r_4 and r_5 being known, the distances of the poles of the fourth and fifth circles from A, B, and C are known, and that consequently the positions of these two circles are determined.

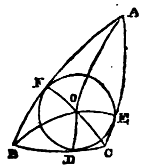
[SECOND SOLUTION.—*Mr. Philip Beecroft, Hyde, Cheshire.*]

Before proceeding to the properties of four circles in mutual contact on a sphere, it will be necessary to establish the following theorem.

PROP. A.

The sum of the rectangles of the cotangents of the radii of three circles on a sphere in mutual contact, is equal to the square of the cosecant of the radius of a circle passing through their points of contact.

CASE I. Let A, B, C be the centres of three circles on a sphere touching each other externally in the points D, E, F,* and O the centre of a circle passing through their points of contact. Suppose great circles to pass through every two of the points A, B, C, D, E, F, O; then the great circles BC, CA, AB, will pass respectively through the points D, E, F, since a great circle passing through the centres of two circles in contact on a sphere will pass through their point of contact; and since



$$AE=AF,\quad BF=BD,\quad CD=CE,$$

it is evident that the circle DEF is inscribed in the triangle ABC, and therefore OD, OE, OF are perpendiculars to BC, CA, AB, respectively.

Denote the radii of the circles whose centres are A, B, C, O, by $\rho_1, \rho_2, \rho_3, r$ respectively; then from the usual equations to a right angled spherical triangle, we have

$$\sin r\cot\rho_1=\cot AOF,\ \sin r\cot\rho_2=\cot BOD,\ \sin r\cot\rho_3=\cot COE\ldots.(1)$$

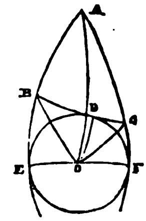
* The positions of the centres of the circles radii ρ_1, ρ_2, ρ_3, and the points D, E, F, through which they pass two and two, are only given, and the circles themselves are omitted in the diagram, as the figures are thus simplified, and as easy to comprehend as if the whole had been drawn.

Multiplying each two of these equations together, and adding, we get

$$\begin{aligned}&\sin^2 r(\cot\rho_1 \cot\rho_2 + \cot\rho_2 \cot\rho_3 + \cot\rho_3 \cot\rho_1)\\ &= \cot AOF \cot BOD + \cot BOD \cot COE + \cot COE \cot AOF\\ &= 1, \text{ since } AOF + BOD + COE = \pi,\end{aligned}$$

hence, $\operatorname{cosec}^2 r = \cot\rho_1 \cot\rho_2 + \cot\rho_2 \cot\rho_3 + \cot\rho_3 \cot\rho_1$.

CASE II. If the circle centre A touch the circles whose centres are B, C internally, as in the annexed figure. It will be seen in the same manner as above that the circle DEF will in this case touch the sides of the triangle ABC, but it will be on the side of BC opposite to A. Continuing the above notation, the equations (1) will also exist here, and



$$2AOF = AOE + AOF = FOD + DOE = 2BOD + 2COE,$$

$$\text{or, } AOF = BOD + COE;$$

$$\therefore \cot AOF = \cot(BOD + COE) = \frac{\cot BOD \cot COE - 1}{\cot BOD + \cot COE};$$

and substituting from (1), we get

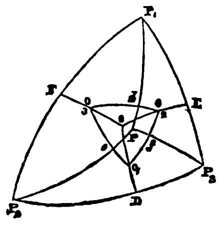
$$\sin r \cot\rho_1 = \cot AOF = \frac{\sin^2 r \cot\rho_2 \cot\rho_3 - 1}{\sin r\,(\cot\rho_2 + \cot\rho_3)};$$

from which we have, $\operatorname{cosec}^2 r = \cot\rho_2 \cot\rho_3 - \cot\rho_1 \cot\rho_2 - \cot\rho_1 \cot\rho_3$.

The last case shows that any properties expressed for circles touching each other externally may be generalized for any other position in which they may touch, by treating the radius of any circle touching the others internally as a negative quantity in the same expression; on which account I shall consider in the following investigations, all the circles as touching externally the circles with which they may have contact, as with this consideration, the expressions deduced will be general for every position of the circles.

ON FOUR CIRCLES IN MUTUAL CONTACT ON A SPHERE.

In a manner exactly similar to the demonstration of Prop. I. of the "Properties of Circles in Mutual Contact," in the Appendix to the Lady's and Gentleman's Diary for 1842, it may be shown that if four circles whose centres are O, O_1, O_2, O_3 be described on a sphere to touch each other mutually in the points D, E, F, *d*, *e*, *f*; four other circles whose centres are P, P_1, P_2, P_3 may be described to touch each other in the same points (D, E, F, *d*, *e*, *f*); each circle in one set of circles in mutual contact passing through the points of contact of three circles in the other set.*



Let r, r_1, r_2, r_3 be the radii of the circles centres O, O_1, O_2, O_3 respectively, and ρ, ρ_1, ρ_2, ρ_3 those of the circles whose centres are P, P_2, P_2, P_3 respectively.

Then from Prop. A we have the equations

$$\left.\begin{aligned}\cot^2 r &= \cot\rho_1 \cot\rho_2 + \cot\rho_2 \cot\rho_3 + \cot\rho_3 \cot\rho_1 - 1\\ \cot^2 r_1 &= \cot\rho \cot\rho_2 + \cot\rho \cot\rho_3 + \cot\rho_2 \cot\rho_3 - 1\\ \cot^2 r_2 &= \cot\rho \cot\rho_1 + \cot\rho \cot\rho_3 + \cot\rho_1 \cot\rho_3 - 1\\ \cot^2 r_3 &= \cot\rho \cot\rho_1 + \cot\rho \cot\rho_2 + \cot\rho_1 \cot\rho_2 - 1\end{aligned}\right\} \quad \ldots\ldots\ldots(1)$$

* The circles DEF, D*ef*. *d*E*f*, *de*F, *def*, F*d*E, E*f*D, D*e*F, centres O, O_1, O_2, O_3, P, P_1, P_2, P_3, are not drawn in the figure, but must be understood.

$$\left.\begin{aligned}\cot^2\rho &= \cot r_1 \cot r_2 + \cot r_2 \cot r_3 + \cot r_3 \cot r_1 - 1\\ \cot^2\rho_1 &= \cot r \cot r_2 + \cot r \cot r_3 + \cot r_2 \cot r_3 - 1\\ \cot^2\rho_2 &= \cot r \cot r_1 + \cot r \cot r_3 + \cot r_1 \cot r_3 - 1\\ \cot^2\rho_3 &= \cot r \cot r_1 + \cot r \cot r_2 + \cot r_1 \cot r_2 - 1\end{aligned}\right\}\ldots\ldots\ldots(2)$$

Hence, adding the first members in (1) to the last in (2), and the first members in (2) to the second in (1); we have evidently

$$(\cot r+\cot r_1+\cot r_2+\cot r_3)^2 = (\cot\rho+\cot\rho_1+\cot\rho_2+\cot\rho_3)^2\ldots\ldots(3)$$

or, $\cot r+\cot r_1+\cot r_2+\cot r_3 = \cot\rho+\cot\rho_1+\cot\rho_2+\cot\rho_3\ldots\ldots(4)$

From (1), $-\cot^2 r+\cot^2 r_1+\cot^2 r_2+\cot^2 r_3 = 2\cot\rho(\cot\rho_1+\cot\rho_2+\cot\rho_3)-2$

$$= 2\cot\rho(\cot r+\cot r_1+\cot r_2+\cot r_3-\cot\rho)-2, \text{ by } (4)$$

or

$$\begin{aligned}2\cot\rho(\cot r+\cot r_1+\cot r_2+\cot r_3) &= -\cot^2 r+\cot^2 r_1+\cot^2 r_2+\cot^2 r_3+2(\cot^2\rho+1)\\ &= -\cot^2 r+\cot^2 r_1+\cot^2 r_2+\cot^2 r_3\\ &\quad+2(\cot r_1\cot r_2+\cot r_2\cot r_3+\cot r_3\cot r_1), \text{ by } (2)\\ &= (\cot r+\cot r_1+\cot r_2+\cot r_3)(-\cot r+\cot r_1+\cot r_2+\cot r_3)\end{aligned}$$

$$\left.\begin{aligned}\therefore\ 2\cot\rho &= -\cot r + \cot r_1 + \cot r_2 + \cot r_3\\ \text{Similarly,}\quad 2\cot\rho_1 &= \cot r - \cot r_1 + \cot r_2 + \cot r_3\\ 2\cot\rho_2 &= \cot r + \cot r_1 - \cot r_2 + \cot r_3\\ 2\cot\rho_3 &= \cot r + \cot r_1 + \cot r_2 - \cot r_3\end{aligned}\right\}\ldots\ldots\ldots(5)$$

Again from (5, 2) we have

$$(-\cot r+\cot r_1+\cot r_2+\cot r_3)^2 = 4\cot^2\rho = 4(\cot r_1\cot r_2+\cot r_2\cot r_3+\cot r_3\cot r_1-1)$$

or, $\cot^2 r+\cot^2 r_1+\cot^2 r_2+\cot^2 r_3+4 = 2\cot r_1\cot r_2+2\cot r_2\cot r_3+2\cot r_3\cot r_1$

$$+2\cot r\cot r_1+2\cot r\cot r_2+2\cot r\cot r_3\ldots(6)$$

And since $\cot^2 r+1=\operatorname{cosec}^2 r$, *etc.*, we get

$$\operatorname{cosec}^2 r+\operatorname{cosec}^2 r_1+\operatorname{cosec}^2 r_2+\operatorname{cosec}^2 r_3 = 2\cot r_1\cot r_2+2\cot r_2\cot r_3+2\cot r_3\cot r_1$$

$$+2\cot r\cot r_1+2\cot r\cot r_2+2\cot r\cot r_3\ldots(7)$$

From (6) we get, by transposition,

$$\begin{aligned}&\cot^2 r-2\cot r(\cot r_1+\cot r_2+\cot r_3)\\ &= 2\cot r_1\cot r_2+2\cot r_2\cot r_3+2\cot r_3\cot r_1-\cot^2 r_1-\cot^2 r_2-\cot^2 r_3-4,\\ &\therefore \cot r = \cot r_1+\cot r_2+\cot r_3\pm 2\sqrt{(\cot r_1\cot r_2+\cot r_2\cot r_3+\cot r_3\cot r_1-1)}\\ &\qquad = \cot r_1+\cot r_2+\cot r_3\pm 2\cot\rho\ldots\ldots\ldots\ldots(8)\end{aligned}$$

This result agrees with the first in (5), and is a solution of the question, the radii r_1, r_2, r_3 representing those of the three given circles in mutual contact; the upper sign taking place in the double sign $\pm$ when r denotes the radius of the lesser of the two circles that can be described to touch the given three, and the lower sign when r denotes the radius of the greater one. When the circle radius r is touched internally by the given three, r must be considered negative in (8), and then

$$-\cot r = \cot r_1+\cot r_2+\cot r_3-2\cot\rho,$$

or in this case, $\cot r = -\cot r_1-\cot r_2-\cot r_3+2\cot\rho\ldots\ldots\ldots(9)$

If any one of the three given circles, as that radius r_1 is touched internally by the other two, we should then have

$$\cot r = -\cot r_1+\cot r_2+\cot r_3\pm 2\cot\rho\ldots\ldots(10)$$

Add together the squares of the equations in (5), and we have

$$\cot^2 r+\cot^2 r_1+\cot^2 r_2+\cot^2 r_3 = \cot^2\rho+\cot^2\rho_1+\cot^2\rho_2+\cot^2\rho_3\ldots(11)$$

From (4, 11) we have this remarkable theorem :

The sum and the sum of the squares of the cotangents of the radii of four circles on a sphere in mutual contact, are respectively equal to the sum and the sum of the squares of the cotangents of the radii of the four other circles that may be described on the same sphere, to touch each other mutually in the same points as those in which the first four touch each other.

The equation (7) to four circles on a sphere in mutual contact, may be thus expressed :

The sum of the squares of the cosecants of the radii of four circles described on a sphere in mutual contact, is equal to twice the sum of the rectangles of the cotangents of these radii.

Many other neat properties may be easily obtained from equations (5, 8, 9, 10), which are left for the student's investigation.

Mr. Weddle, of Newcastle-on-Tyne, favored us with an elegant solution, analogous to the preceding, and after showing that there are two circles which touch three circles in mutual contact on the surface of a sphere, he adds the following properties :

Let r and R be the radii of these two circles, then by (8)

$$\cot r + \cot . R = 2\cot r_1 + 2\cot r_2 + 2\cot r_3 \ldots\ldots\ldots\ldots (a)$$

If the centres of the three circles be on the same great circle, we have $r_1 + r_2 + r_3 = \pi$, and (8) reduces to

$$\cot r = \cot r_1 + \cot r_2 + \cot r_3 \ldots\ldots\ldots\ldots\ldots\ldots (b)$$

If in (8) we suppose the circles whose radii are r, r_1, r_2 to be touched by a great circle, we must have $r_3 = \frac{\pi}{2}$,

$$\therefore \quad \cot r = \cot r_1 + \cot r_2 \pm 2\sqrt{\cot r_1 \cot r_2 - 1} \ldots\ldots\ldots\ldots (c)$$

which determines the radii of the circles touching two circles and the tangent great circle.

If in the last equation we further suppose $r_1 + r_2 = \frac{\pi}{2}$, we shall have

$$\tan r = \tfrac{1}{2}\sin 2r_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d)$$

Eq. (8) gives the magnitude of the required circle, and to determine its position, let r', θ be the spherical polar co-ordinates of the centre, O_1 being the origin, and O_1O_2 the axis ; then

$$r' = r_1 + r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (e)$$

and by the usual trigonometrical formula, we get

$$\cot^2\frac{\theta}{2} = \sin^2 r_1 (\cot r \cot r_1 + \cot r \cot r_2 + \cot r_1 \cot r_2 - 1) = \sin^2 r_1 \cot^2 \rho_3$$

$$\therefore \cot\frac{\theta}{2} = \sin r_1 \cot\rho_3 = \frac{\sin r_1}{2}(\cot r + \cot r_1 + \cot r_2 - \cot r_3) \ldots\ldots (f)$$

⁎ Though we have inserted solutions of the greater part of the questions proposed in our first number, our plan admits of our giving additional ones that may be sent us, provided they be distinguished by originality and elegance, at any future time.

Some letters arrived relative to the questions too late to be used, as the copy was in the hands of the printer. We especially refer to the letter of Mr. Nicholas Smyth, of Galway, containing solutions of 1, 2, 3, 4, 5, 7, to some of which we may possibly have an opportunity of recurring hereafter.

MATHEMATICAL EXERCISES—(continued.)

8.—*By φ.*

[illegible] circumference of a circle be divided into six equal parts i[illegible] $A_1 \ldots A_6$, and if A_1A_3 be joined intersecting the radius [illegible] A_2 joined meeting OA_1 in B_2; B_2A_3 joined cutting OA_4 in B_3; [illegible] OB_2, OB_3, *etc.* are respectively the half, third, fourth, *etc.* p[illegible] radius.

9.—*Mr. Fenwick.*

From a point P without a circle draw the tangents PA, PB, a[illegible] [illegible] PC to the concave circumference: then the tangent at C, a pe[illegible] [illegible] PC, and the line which joins the middle points of PA, [illegible] the same point.

10.—*E. I. F., London.*

Given the system of equations,

$$\begin{aligned} a &= a \\ a + n &= ab \\ a + 2n &= ab^2c \\ a + 3n &= ab^3c^2d \\ &\vdots \end{aligned}$$

Find the factors, the law of successive developement, and the mu[illegible] the two series.

11—*Mr. Thomas Dobson, Totteridge, Herts.*

A, B, C are three given points in a straight line, of which C i[illegible] a given circle, and a straight line is drawn through B inte[illegible] circle in the points D, E; find the position of this line when th[illegible] appears a maximum to a spectator at A.

12.—*W. F., Durham.*

The six co-ordinate axes of any two systems of rectangular co-[illegible] (the origin being the same) lie in a conical surface of [illegible] degree: also the six co-ordinate planes touch a conical su[illegible] [illegible] degree.

13.—*Mr. R. H. Wright, London.*

If S be the focus of an ellipse and the centre of force, P*p* a ch[illegible] then the time of a body revolving from P to *p* is

$$t = \frac{a^{\frac{3}{2}}}{u^{\frac{1}{2}}}(2m - \sin 2m);$$

where $\tan m = \dfrac{\tan e}{\tan \theta}$, $\angle$ ASB $= \theta$, and eccentricity $= \cos e$.

14.—*Mr. Rutherford.*

Let r, r_1, r_2, r_3 be the radii of four circles in mutual contact [illegible] or in a plane, of which the circle radius r is touched exte[illegible] three; then if d and d_1 are the respective distances of th[illegible] circles radii r and r_1 from the great circle or line joining th[illegible] circles radii r_2 and r_3, we shall have

ON THE SPHERE.	ON THE PLANE.
$\dfrac{\sin r}{\sin d} - \dfrac{\sin r_1}{\sin d_1} = 2$	$\dfrac{r}{d} - \dfrac{r_1}{d_1} =$ [illegible]

15.—*Mr. Philip Beecroft, Hyde, Cheshire.*

[illegible] sum of the squares of the cosecants of the perpendicul[illegible] of a spherical triangle upon the opposite sides, is equal [illegible] sum of the squares of the cosecants of the radii of th[illegible] the sides of the triangle.

A LOCUS: FROM LHUILLIER'S POLYGONOMETRIE.

The following locus has been determined by Lhuillier in a very complex manner. As it discloses some elegant properties of the centroïd of a physical or geometrical system, a simple and direct investigation will not be without interest.

There are given n lines SA_1, $SA_2 \ldots . SA_n$ in magnitude and position, and Q^2 is a given magnitude: it is required to find the locus of the point Y, such that when perpendiculars YB_1, $YB_2, \ldots . YB_n$ are drawn to the given lines, we shall have

$$SA_1 . YB_1 + SA_2 . YB_2 + \ldots\ldots + SA_n . YB_n = Q^2.$$

Take S as origin of rectangular co-ordinates, and let the axis of x pass through the *centroïd* Z; denote the angles made with this axis by the given lines by a_1, $a_2, \ldots . a_n$; and the distances of A_1, $A_2, \ldots . A_n$ from S by $r_1, r_2, \ldots . r_n$.

Then we shall have from the known properties of the *centroïd*,

$$r_1 \cos a_1 + r_2 \cos a_2 + \ldots . + r_n \cos a_n = SZ \ldots\ldots\ldots\ldots\ldots (1)$$

$$r_1 \sin a_1 + r_2 \sin a_2 + \ldots . + r_n \sin a_n = 0 \ldots\ldots\ldots\ldots\ldots (2)$$

Then the equations of $SA_1, \ldots .$, and the lengths of $YB_1, \ldots .$ from xy, will be as follow:

Equations of Lines.	*Lengths of Perpendiculars.*
$(SA_1) \ldots . y = x \tan a_1$	$YB_1 = y \cos a_1 - x \sin a_1$
$(SA_2) \ldots . y = x \tan a_2$	$YB_2 = y \cos a_2 - x \sin a_2$
…………………	…………………
$(SA_n) \ldots . y = x \tan a_n$	$YB_n = y \cos a_n - x \sin a_n$

Forming from these elements the equation of the problem, we get

$$y(r_1 \cos a_1 + r_2 \cos a_2 + \ldots .) - x\,(r_1 \sin a_1 + r_2 \sin a_2 + \ldots .) = Q^2 . . (3)$$

which is the equation of a straight line.

But in virtue of (1, 2) this becomes simply

$$y = \frac{Q^2}{SZ} .$$

whence the straight line is parallel to the axis of x, and the following construction gives the locus sought.

Find the *centroïd* Z of the given system of points A_1, $A_2, \ldots . A_n$; and join SZ. Find a third proportional to SZ and Q; and draw a line parallel to SZ at the distance of this third proportional.

Scholium 1.—The more general problem, viz. when the lines do not meet in one point, may be investigated in the same manner, and the locus will still be a straight line. For the expression becomes in this case,

$$y\,(r_1 \cos a_1 + \ldots .) - x\,(r_1 \sin a_1 + \ldots .) = (r_1 b_1 \cos a_1 + \ldots .) + Q^2.$$

where r_1, $r_2, \ldots .$ are given portions of lines given in position, and b_1; $b_2, \ldots .$ are the constants which, with a_1, $a_2, \ldots .$ determine the position of those lines. Even for the triangle, Lhuillier's investigation is extremely intricate: but the co-ordinate method is simple and concise for the most general case.

Scholium 2.—Any properties of the *centroïd*, or of the equations by which it is expressed, will always be acceptable from our correspondents: and when we are in possession of a sufficient number to form a paper on the subject, we shall give them, with the names of their authors, in the manner of the *Horæ Geometricæ* in the *Ladies' Diary*.

27th *Jan.* 1844. D. V. S.

ON THE CONDITIONS OF EQUILIBRIUM

BETWEEN FORCES WHICH ACT UPON FOUR SPHERES IN MUTUAL CONTACT THREE AND THREE.

[*From a Correspondent.*]

Let A, B, C, D (the student will easily sketch the fig.) be the centres of the four spheres designated by their respective radii r_1, r_2, r_3, r_4; and E, F, G, H their four points of contact in order, so that AH $=$ AE $= r_1$, BE $=$ BF $= r_2$, *etc.* The points E, F, G, H lie in the common normals of the spheres r_1r_2, r_2r_3, r_3r_4, r_4r_1, respectively. The mutual pressures of the spheres r_1r_2 are in the lines EA, EB, denote their intensity by a_1. Similarly, denote by a_2, a_3, a_4, those at F, G, H.

Now the forces acting on the sphere r_1 with the pressures a_1, a_4, which it receives at E and H in the directions EA, HA, must be in equilibrium, and since these directions meet in the point A, the forces which act on r_1 must have a resultant AP $= p$. This resultant being in equilibrium with a_1, a_4, must act in the plane DAB, and by principles of statics

$$a_4 : a_1 : p :: \sin \text{PAB} : \sin \text{PAD} : \sin \text{DAB} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly, if q, r, s be the corresponding resultants through B, C, D, and BQ, CR, DS their directions, we have

$$a_1 : a_2 : q :: \sin \text{QBC} : \sin \text{QBA} : \sin \text{ABC} \text{ (in the plane ABC)} \ldots (2)$$

$$a_2 : a_3 : r :: \sin \text{RCD} : \sin \text{RCB} : \sin \text{BCD} \text{ (in the plane BCD)} \ldots (3)$$

$$a_3 : a_4 : s :: \sin \text{SDA} : \sin \text{SDC} : \sin \text{CDA} \text{ (in the plane CDA)} \ldots (4)$$

By the elimination of a_1, a_2, a_3, a_4 from (1, 2, 3, 4) we get the equation of condition

$$\frac{\sin \text{PAB}}{\sin \text{PAD}} \cdot \frac{\sin \text{QBC}}{\sin \text{QBA}} \cdot \frac{\sin \text{RCD}}{\sin \text{RCB}} \cdot \frac{\sin \text{SDA}}{\sin \text{SDC}} = 1 \ldots\ldots (\alpha)$$

And the mutual relation of the forces p and q is

$$p : q :: \frac{\sin \text{DAB}}{\sin \text{DAP}} : \frac{\sin \text{ABC}}{\sin \text{QBC}} \ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

In a similar way the ratios of $q : r$ and $r : s$ are obtained.

Hence the four following are the conditions of equilibrium:

(1.) The forces acting upon each of the four spheres must have a resultant passing through the centre of each sphere respectively.

(2.) Each of these resultants must be in the same planes as the centres of the three adjacent spheres.

(3.) There must be the relation (α) between the directions of the resultants.

(4.) There must be the relation (β) between their intensities.

Jan. 1844. ϕ.

MATHEMATICAL NOTES.

I. *On Guldin's Rule.*—The rule of Pappus, commonly called Guldin's rule, may be applied sometimes to find areas or volumes, and sometimes also to determine the position of the centre of gravity.

(1.) Let AC be the base of a triangle, and BD a perpendicular from the opposite angle B upon AC or AC produced, and let the triangle ABC revolve about AC as an axis, and describe a solid whose volume is $= \frac{\pi}{3}$ AC.BD^2. Now the area of the triangle is $\frac{1}{2}$ AC.BD, and the path of 'tre of gravity is $= 2\pi y_0$; therefore, by Guldin's rule,

$$2\pi y_0 . \frac{\text{AC.BD}}{2} = \frac{\pi}{3}\text{AC.BD}^2 \qquad \text{or, } y_0 = \frac{1}{3} \text{BD}.$$

The centre of gravity of the triangle ABC is therefore situated in a line parallel to AC, at a distance of $\frac{1}{3}$ BD from AC, and the point of intersection of this line with the line drawn from B to the middle of AC is the centre of gravity of the triangle.

(2.) Let OM be a parabola, OP the abscissa, and MP the ordinate.

The equation of the parabola is $y^2 = px$.

If x_0, y_0 are the co-ordinates of the centre of gravity of the area OPM; then it is known that

$$x_0 = \frac{3x}{5}, \text{ and } y_0 = \frac{3y}{8}.$$

Now the area OPM $= \frac{2}{3} xy$; because OP $= x$, and if the surface OPM revolve about OP, the path of its centre of gravity is $= 2\pi . \frac{3}{8} y = \frac{3}{4} \pi y$, and this multiplied by the area of the revolving surface $\frac{2}{3} xy$ gives $\frac{1}{2} \pi xy^2$, or $\frac{1}{2} \pi px^2$, for the volume of the paraboloid of revolution.

a.

II. A triangle is formed by joining the bisections of the sides of a given triangle, another by joining the bisections of the sides of the new triangle, and so on; r, r_n, are the respective radii of the circles *inscribed* in the original and the n^{th} of the new triangles; R, R_n, are the same for the *circumscribed circles*: then

$$r_n = \frac{r}{2^n}, \text{ and } R_n = \frac{R}{2^n}.$$

The sides of the first new triangle are respectively half those of the original one; this is well known. Hence, denoting the sides of the original triangle by a, b, c, and those of the 1st, 2nd, ... n^{th}, of the new triangles by a_1, b_1, c_1; a_2, b_2, c_2; .... a_n, b_n, c_n; we have

$$a_1 = \frac{1}{2} a; \; b_1 = \frac{1}{2} b; \; c_1 = \frac{1}{2} c.$$

Similarly, $a_2 = \frac{1}{2} a_1 = \frac{1}{2^2} a$; $b_2 = \frac{1}{2} b_1 = \frac{1}{2^2} b$; $c_2 = \frac{1}{2} c_1 = \frac{1}{2^2} c$;

$$a_3 = \ldots\ldots \frac{1}{2^3} a; \; b_3 = \ldots\ldots \frac{1}{2^3} b; \; c_3 = \ldots\ldots \frac{1}{2^3} c;$$

....

$$a_n = \ldots\ldots \frac{1}{2^n} a; \; b_n = \ldots\ldots \frac{1}{2^n} b; \; c_n = \ldots\ldots \frac{1}{2^n} c.$$

$$\therefore \; 2s_n = \frac{1}{2^n}(a+b+c) = \frac{1}{2^n} . 2s \quad \therefore \quad s_n = \frac{1}{2^n} s.$$

$$\therefore s_n - a_n = \frac{1}{2^n}(s-a); \; s_n - b_n = \frac{1}{2^n}(s-b); \; s_n - c_n = \frac{1}{2^n}(s-c)$$

$$\therefore \; s_n (s_n - a_n)(s_n - b_n)(s_n - c_n) = \frac{1}{2^{4n}} s(s-a)(s-b)(s-c),$$

$$\text{and, } r^2_n = \frac{s_n(s_n - a_n)(s_n - b_n)(s_n - c_n)}{s^2_n} = \frac{s(s-a)(s-b)(s-c)}{2^{2n} s^2} = \frac{r^2}{2^{2n}};$$

$$\therefore \; r_n = \frac{r}{2^n}.$$

The other relation is established in a similar way.

γ.

ON ALGEBRAIC TRANSFORMATION,

AS DEDUCIBLE FROM FIRST PRINCIPLES; AND CONNECTED WITH CONTINUOUS APPROXIMATION AND THE THEORY OF FINITE AND INFINITESIMAL DIFFERENCES: INCLUDING SOME NEW MODES OF NUMERICAL SOLUTION.

[*By the late W. G. Horner, Esq.*]

Read before the Royal Society, June 19, 1823.

[No mathematician has contributed so largely as Mr. Horner to the improvement of the problem of the numerical solution of equations; and perhaps no one has left the world with cause so just for considering himself treated with unmerited neglect, and even personal hostility, in connexion with his researches. It is not difficult to account for this: and it reads a lesson to every mathematician;—that to obtain the reputation to which his discoveries really entitle him, he must not only make those discoveries, but also, somehow or other, connect himself with a " class," so as to secure an *enforcement* of his claims upon the public mind. A good deal of intrigue and a little science will effect more in the way of cotemporary reputation, than much science, without foreign aid, can ever effect for him. This, indeed, is a melancholy truth: nevertheless, a truth it is.

Mr. Horner's first paper on equations was printed in the Philosophical Transactions for 1819; and Mr. Horner has often stated to me, that much demur was made to the insertion of it in that publication. It was, indeed, owing more to the influence and earnestness of Mr. Davies Gilbert, than to any respect to the author, his subject, or his mode of treating it, that that honor was accorded to him. The *elementary character* of the subject was the professed objection: his recondite mode of treating it, was the professed passport for its admission.

That paper has been reprinted in the *Ladies' Diary* for 1838; and most of our readers are doubtless acquainted with it. That the mode in which it is drawn up is in one respect fortunate, there can be no doubt, since that finally secured its publication: whilst, on the other hand, it may be considered unfortunate by its requiring so much higher mathematical learning to understand the reasoning, than the nature of the inquiry itself renders desirable. Mr. Horner was himself so sensible of this objection, that he immediately attempted a simplification of the principles. The consequence of this attempt, was the paper now about to be submitted to the public for the first time, after lying more than twenty years altogether unknown. The first objection made to it was its *length:* the author curtailed it, under the impression made upon his mind, that it would be printed when so altered: the end of all his labour was,—that it was " deposited in the archives."

That the quality which rendered this paper objectionable in the Philosophical Transactions will be a recommendation of it to most of the readers of the Mathematician, there can be no doubt: but even as a general method of investigation; as evidences of the high mathematical powers of the author; as involving much difficult, delicate, and original thinking;—it would not be a discredit even now, to any author, or to any publication. The paper, indeed, is full of original views: and it will be seen, too, that even after such a lapse of time, every part is instructive.] T. S. DAVIES.

1. When the principle of Continuous Approximation was first developed, in the Phil. Trans. for 1819, no means of arriving at the same results had occurred to me, which were at the same time so simple and so unrestricted as those which are detailed in the paper alluded to, and which actually conducted me to the discovery.

Subsequent reflection has, however, convinced me that even the incidental use of *figurate series* must be regarded as foreign to the question. Designating merely the combined effect of elements, which it is the calculator's purpose to introduce separately, they are alike absent from the premises, and from those practical conclusions in which our researches terminated. What is of still greater consequence, while engaged on exterminating them, the investigation is necessarily confined to the particular case of which they happen to be the symbols. So that it is not at all surprising that a much more extensive view of the subject may be obtained by following out a very elementary idea.

2. The remark I am about to make may appear, indeed, at first sight, too trivial to be dwelt upon. If the formula

$$A_{n+1}\, y^{n} + B_{n}\, y^{n-1} + \dots\dots + K_4 y^3 + L_3 y^2 + M_2 y + N_1 \dots (1)$$

be divided by y as many times in succession as are represented by any one of the subscribed exponents, the several remainders will be N_1, M_2, L_3, *etc.*, ending with that under which the selected exponent appears. Yet if we conceive y to be only a concise expression for $x - r$, and the formula (1) to be only a varied enunciation of

$$A_0 x^n + B_0\, x^{n-1} + \dots\dots\dots K_0\, x^3 + L_0\, x^2 + M_0\, x + N_0 = fx \dots (2)$$

we find in this perfect truism, the real germ of that species of transformation on which approximate solution depends.

3. In fact, if equation (2) be divided continually by $x - r$, the successive remainders will be N_1, M_2, L_3, *etc.*, the co-efficients of (1); and no objection lies against this as a practical mode of transforming, except the tedious and unharmonising nature of the ordinary process of algebraic division. Our first care, therefore, should be directed to the removal of this obstacle.*

4, 5. [These sections are devoted to an investigation of the principles of *Synthetic Division*, and to a statement of the rule of operation deduced

* In the contracted copy of the MS. the following, merely, is substituted for the sections (5, 6, 7, 8): and as it does really point out the original course pursued by Mr. Horner in the discovery of the *Synthetic Division*, it is worthy of preservation.

"To remove this obstacle, we have only to exchange the *divisor* for a *scale of relation*, as in recurring series. The work will most naturally assume the arrangement proper to algebraic multiplication, which is well adapted to numerical operation.

"In the instance before us, the scale of relation is r. Leaving the powers of x to be mentally annexed to these coefficients, and indicating in the margin the number of times that r must be used to arrive at the remainder, the continual division of Eq. (2) by $x - r$ will assume this form:

	A_0	B_0	C_0	....	K_0	L_0	M_0	N_0
	0	$A_1 r$	$B_1 r$	....	$H_1 r$	$K_1 r$	$L_1 r$	$M_1 r$
r^n,	A_1	B_1	C_1	....	K_1	L_1	M_1	N_1
	0	$A_2 r$	$B_2 r$	....	$H_2 r$	$K_2 r$	$L_2 r$	
r^{n-1},	A_2	B_2	C_2	....	K_2	L_2	M_2	
	0	$A_3 r$	$B_3 r$	....	$H_3 r$	$K_3 r$		
r^{n-2},	A_3	B_3	C_3	....	K_3	L_3		
		etc.		*etc.*	*etc.*			

"This process is in substance and in practice, the theorem of art. 14, in my former paper, and the improved method of arts. 16 and 18 flows directly from it. What has been here advanced may serve as a new investigation, and the simplest, perhaps, that can be given of those theorems."

from it. The method itself was published in a note to the reprint of Mr. Horner's first paper in the Ladies' Diary, 1838: and as a different investigation and a complete statement of the rule are given in the present number of the Mathematician, pp. 74—76, it is unnecessary to print it here. The example, however, is retained, on account of its subsequent bearing.]

Ex. I. Multiply the several terms of $x^3 - bx - c = 0$ by their indices, and divide the result by the original quantity.

Here $3x^3 - bx$ is to be divided by $x^3 - bx - c$; or

$3 - bx^{-2}$ by $1 - bx^{-2} - cx^{-3}$. Whence

$$\begin{array}{r|l}
1 & 3 + 0 - b \\
+ b & \qquad\quad 3b + \;0\; + 2b^2 + 3bc + 2b^3 + 5b^2c \ldots\ldots \\
+ c & \qquad\qquad\quad 3c + \;0\; + 2bc + 3c^2 + 2b^2c + 5bc^2 \ldots \\
\hline
 & 3 + 0 + 2b + 3c + 2b^2 + 5bc + (2b^3 + 3c^2) + 7b^2c + \ldots
\end{array}$$

the successive powers of x^{-1} being understood.

6. An observation occurs here which, I think, should not be suppressed, as it furnishes a strong collateral plea for introducing this mode of Division among the elementary rules of Algebra. It is this:—The enunciation of this example contains in a few words and devoid of technicality, the most essential part of the application of Recurring Series, to Equations, and the short and universal rule which guides the work is more apprehensible than the notion of a Scale of Relation. Mathematicians need not be reminded that the coefficients 3, 0, $2b$, *etc.*, found above, are equal to the sums of the 0, 1, 2, *etc.* powers of the roots of the given equation; so that the more advanced terms differ less and less from the pure powers of the principal root, and hence various methods readily suggest themselves for approximating to that root.—(See *Newton, Univ. Arith.*: *Euler, Anal. Inf. cap.* 17: *Algebra, art.* 792: *Arbogast* § 335: *Lagrange, Eq. Num., note* 6.)

7. Ex. II. Divide formula 2 by $x - r$.

The first coefficient of the divisor being 1, and the law of the literal parts conspicuous, we shall operate, as before, with the coefficients alone.

$$\begin{array}{lllllll}
\text{To } A_0 & B_0 & C_0 \ldots\ldots\ldots K_0 & L_0 & M_0 & N_0 \\
\text{Add } 0 & A_1r & B_1r \ldots\ldots\ldots H_1r & K_1r & L_1r & M_1r \\
\hline
r^n \ldots\, A_1 & B_1 & C_1 \ldots\ldots\ldots K_1 & L_1 & M_1 & N_1
\end{array}$$

Hence the quotient is

$$A_1 x^{n-1} + B_1 x^{n-2} + C_1 x^{n-3} + \ldots + K_1 x^2 + L_1 x + M_1 + \frac{N_1}{x - r}.$$

In this example the division is stopped at that term which is ordinarily called the remainder; namely, that which succeeds the absolute term of the quotient, or in which the index of x first becomes negative: and agreeably to the tenor of *Art.* 3, we might hence deduce a ready method of transformation, the identical method, in fact, which under a different notation, appears in *Art.* 14 of my former paper. But a more fitting place will be found for it in the sequel, as a particular Corollary of a general principle. (*Arts.* 21, 22.)

8. In the mean time, to avoid the perplexity that would result from ...ining the notion of remainders whilst the literal parts are suppressed, we ...epropose the last example with an addition, suggested indeed by that ...le, and which will insure the termination of the work.

Ex. III. Multiply formula 2 by y and divide by $x - r = y$. That is, Divide $A_0 x^n y + B_0 x^{n-1} y + \dots K_0 x^3 y + L_0 x^2 y + M_0 xy + N_0 y$ by $x - r = y$.

Here, if we suppose the divisor $x - r$ to be employed as far as x appears in the dividend, and then to be exchanged for its equivalent, y, which being a simple term, furnishes no addend to be carried forward, the work will necessarily terminate where the original expression ends. For the rest, the coefficients will be transformed exactly as in Ex. ii., and the quotient will be

$$A_1 x^{n-1} y + B_1 x^{n-2} y + \dots + K_1 x^2 y + L_1 xy + M_1 y + N_1 = fx,$$

a function which may be properly called a *transformée* of the given one, since it differs from this in form only, whilst its value and dimensions remain unaltered. To distinguish it from similar functions of a single unknown quantity, it may be termed a *mixed transformée*.

9. Following the indications of this example, I propose to consider TRANSFORMATION, not, perhaps, in the most extensive acceptation of the term, yet in a very comprehensive one, as flowing from the following axiom:

Any quantity after being divided by a fraction whose numerator and denominator are equivalent, remains unaltered in its value.

Its ***form*** will, manifestly undergo an alteration, if the denominator be, as in the preceding article, a simple symbol, and its numerator a binomial or other compound quantity.

10. Assuming, therefore, $x_m = x - r_m$, and consequently $= x_p - (r_m - r_p)$, where p admits of all values from 0 to m inclusively, let us examine the effect of dividing any portion of a function of x_p by $\frac{x_p - (r_m - r_p)}{x_m}$. And here we immediately perceive that in the first term to which we apply the operation, one x_p is merely exchanged for an x_m, the product of $r_m - r_p$ by the coefficient of this term being carried forward to augment that of the next inferior power of x_p. We also perceive that this transformation may be carried on from term to term as often, and with as great a variety as the said terms allow the option of values for x_p. And, finally, that we may stop the process at any term whatever, by simply making $p = m$.

From these conditions it follows, that endless diversities of transformation will arise, according to the system by which we select values of p. The best general basis will undoubtedly be laid by adhering strictly to the principle of Eq. iii., in retaining the lowest value of p as long as is practicable, and varying it according to the order of the natural series. Any other useful variety may readily be deduced from this.

11. Let, therefore, the expression for fx (*Art.* 2) be multiplied by x_1 and the result divided by $x - r_1$. Then let this quotient be multiplied by x_2 and divided by $x - r_2$, and so on. During each course of division we have only to recollect that

$$x - r_1 = x_1,$$
$$x - r_2 = x_1 - (r_2 - r_1) = x_2,$$
$$x - r_3 = x_1 - (r_3 - r_1) = x_2 - (r_3 - r_2) = x_3,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

and not to employ a more advanced form of any of these divisors of the unknown term, if an earlier form appears in that term of the dividend which we are actually dividing.

12. The successive transformées will then assume the following forms:

$$A_0 x^n + B_0 x^{n-1} + \dots K_0 x^3 + L_0 x^2 + M_0 x + N_0 = fx$$
$$A_1 x^{n-1} x_1 + B_1 x^{n-2} x_1 + \dots K_1 x^2 x_1 + L_1 x x_1 + M_1 x_1 + N_1 = fx$$
$$A_2 x^{n-2} x_1 x_2 + B_2 x^{n-3} x_1 x_2 + \dots K_2 x x_1 x_2 + L_2 x_1 x_2 + M_2 x_2 + N_2 = fx$$
$$A_3 x^{n-3} x_1 x_2 x_3 + B_3 x^{n-4} x_1 x_2 x_3 + \dots K_3 x_1 x_2 x_3 + L_3 x_2 x_3 + M_3 x_3 + N_3 = fx$$

....

and generally

$$\dots\dots H_m x_{m-3} x_{m-2} x_{m-1} x_m + K_m x_{m-2} x_{m-1} x_m + L_m x_{m-1} x_m + M_m x_m + N_m = fx.$$

Cor. Hence $fx_m = N_m$, or the absolute terms are the same as the results of substituting r_m for x in the given equation.

13. The coefficients, according to the principle of division already laid down, will be successively developed in the following manner:

	A_0	$B_0 \dots\dots$	K_0	L_0	M_0	N_0
	0	$A_1 r_1 \dots\dots$	$H_1 r_1$	$K_1 r_1$	$L_1 r_1$	$M_1 r_1$
r_1^n,	A_1	$B_1 \dots\dots$	K_1	L_1	M_1	N_1
	0	$A_2 r_2 \dots\dots$	$H_2 r_2$	$K_2 r_2$	$L_2 r_2$	$M_2(r_2 - r_1)$
r_2^{n-1},	A_2	$B_2 \dots\dots$	K_2	L_2	M_2	N_2
	0	$A_3 r_3 \dots\dots$	$H_3 r_3$	$K_3 r_3$	$L_3(r_3 - r_1)$	$M_3(r_3 - r_2)$
r_3^{n-2},	A_3	$B_3 \dots\dots$	K_3	L_3	M_3	N_3
	0	$A_4 r_4 \dots\dots$	$H_4 r_4$	$K_4(r_4 - r_1)$	$L_4(r_4 - r_2)$	$M_4(r_4 - r_3)$
r_4^{n-3},	A_4	$B_4 \dots\dots$	K_4	L_4	M_4	N_4

....

and generally,

$\dots K_{m-1}$	L_{m-1}	M_{m-1}	N_{m-1}
$\dots H_m(r_m - r_{m-4})$	$K_m(r_m - r_{m-3})$	$L_m(r_m - r_{m-2})$	$M_m(r_m - r_{m-1})$
$\dots K_m$	L_m	M_m	N_m

14. In this theorem each original coefficient appears as a function of a variable, which receives, either fortuitously or according to any law whatever, the successive increments r_1, $(r_2 - r_1)$, $(r_3 - r_2)$, ... $(r_m - r_{m-1})$; and the progressive transformations manifestly observe the following general law: which is exactly similar to what is observable in factorial combinations. (*Art.* 13.)

At every new transformation the first of the multipliers previously employed is omitted, and the rest, each augmented by the new increment of the variable, retrograde one step: the new increment itself supplying the place of such multipliers as are still deficient.*

The marginal memoranda should never be omitted. They consist of the leading multipliers in the respective courses, with indices annexed, which decrease regularly till they become stationary at 1. The *index* shews how often the *intire multiplier* is to be used; and *the difference between the index and n*, the highest, shews *at what distance back* the increment was introduced, which is to be *first deducted* from the multiplier.

(*To be continued.*)

* A plural expression is adopted to prevent any perplexity when a real increment [su]cceeds one or more zero-increments.

ON THE TRANSFORMATION OF ALGEBRAIC EQUATIONS.

[*James Cockle, B.A., Trin. Col. Cam., of the Middle Temple.*]

1. Before proceeding to the main subject of this paper, there are one or two observations which it would be improper to omit, especially as the fact, which forms the basis of one of them, appears to afford a direct contradiction to the theory which I advanced in the last number of this periodical.

In a solution of a cubic equation, deprived of its second term, which I communicated to the *Philosophical Magazine*,* the given cubic, in x, is supposed to be equivalent to another in $x+p$, of the binomial form; hence it might be supposed that the condition $\pi(x)=0$ was satisfied: such, however, is not the case, for $\pi(x)=-3a$, where a is the coefficient of x in the given equation. This apparent inconsistency with theory arises from the fact, that the quantity p has three pairs of values, each pair corresponding to a different value of x: hence the laws of symmetric formation do not apply to the coefficients of the subsidiary equation, considered as an equation in x.

Another preliminary remark is this,—those cases in which the equations at pages 83 and 84 can be satisfied by $\lambda=0$ and $\lambda'=1$, are cases in which the given equations admit of, what may be termed, *algebraical* solutions, that is to say, in which they may be solved without having any recourse to the theory of equations. Thus, for instance, the solution of a cubic referred to at the bottom of page 84 is an algebraical one. Now, were we able, by any means, to obtain such an algebraical solution of a biquadratic as should reduce it to the form (3), this solution would have an advantage over the ordinary ones, all † of which proceed by reducing the given equation to the less convenient form $(X^2_1+A^2_1)(X^2_2+A^2_2)=0$. It is not impossible that some modification of the assumption made use of at the page last cited may give the required algebraical solution of a biquadratic, even if the particular one there made use of fail in so doing. But to proceed—

2. Of the three points suggested in my communication to the last number of the *Mathematician*, there are two, namely,

(1.) The structure and properties of the functions there denoted by π, and other analogous functions, and

(2) The change of the form of the auxiliary equation, the consideration of which I shall defer till a future occasion. A short notice of the third will be necessary here, as explanatory of the remaining portion of this paper.

In seeking to destroy m consecutive terms of the equation of the n^{th} degree, commencing with the second term, instead of assuming a transformed equation in v, in which those m terms vanish, it is better to assume

$$(v+p)^n+av^{n-m-1}+\text{\&c.}\ldots.+b=0\ldots\ldots\ldots.(8)$$

which can immediately be reduced to the required form. The advantage of this method is twofold; for, first, we diminish the number of equations of condition; and, secondly, we simplify those which remain. If the terms which we wish to destroy are not consecutive, or if, instead of destroying,

* For June, 1843, vol. xxii. This solution is as simple, and not much longer than the ordinary solution of a quadratic. Its limits are the same as those of Cardan's, but I have not had leisure to discuss it by an independent method. Perhaps one of your correspondents will do so?

† This is incidentally shown in my extension of Descarte's method, C.M.J., vol, iii.

we wish to modify them, in any given manner, the above method is still, with slight alteration, applicable.

3.* Let any function, homogeneous and of the a^{th} degree with respect to the $b+1$ quantities $\text{A}\text{A}'\ldots\text{M}$, be divided by M^a, and, denoting by $z_1, z_2 \ldots z_b$ the ratios which $\text{A}, \text{A}' \ldots \text{A}^{(b)}$ respectively bear to M, let the function when so divided be represented by $f^a(b)$. Then $f^2(3)=0$ is equivalent to

$$z_1^2 + (cz_2 + dz_3 + f)\,z_1 + az_2^2 + (ez_3 + g)z_2 + bz_3^2 + hz_3 + i = 0 \ldots (9)$$

add to, and subtract from, this last expression the square of half the coefficient of z_1, and let $h_1 = z_1 + \frac{1}{2}(cz_2 + dz_3 + f)$,

then (9) becomes

$$h_1^2 + \tfrac{1}{4}(4a - c^2)z_2^2 + \tfrac{1}{2}\{(2e - cd)z_3 + 2g - cf\}z_2 +$$
$$\tfrac{1}{4}\{(4b - d^2)z_3^2 + (4h - 2df)z_3 + 4i - f^2\} = 0.$$

Hence, if z_3 be determined from the quadratic

$$(4a - c^2)\{(4b - d^2)z_3^2 + (4h - 2df)z_3 + 4i - f^2\} = \{(2e - cd)z_3 + 2g - cf\}^2,$$

$f^2(3)=0$ will be reduced to $h_1^3 + h_2^2 = 0$, where

$$h_2 = \tfrac{1}{2}(4a - c^2)^{\frac{1}{2}}\, z_2 + \tfrac{1}{4}\text{A},$$

A being a known quantity.

4. Let b be any odd number greater than 3, then, as before,

$f^2(b) = h_1^2 + f^2(b-1)$, where $f^2(b-1)$ is free from z_1; again

$f^2(b-1) = h_2^2 + f^2(b-2)$, where $f^2(b-2)$ is free from z_2,

&c. = &c. Similarly,

$f^2\{b-(b-3)\} = f^2(3) = h_{b-2}^2 + h_{b-1}^2$, z_b being determined by the method given in the last paragraph. Adding these equations, we obtain

$$f^2(b) = \Sigma\,(h^2) \ldots\ldots\ldots\ldots (10)$$

the number of terms in this last expression being $b-1$, which is an *even* number.

5. Hence $f^2(b)$ may, when b is an odd number, be reduced to a simple equation containing $\frac{1}{2}(b+1)$ disposable ratios; for $\Sigma\,(h^2)=0$ may be decomposed into the $\frac{1}{2}(b-1)$ equations

$$h_1^2 + h_2^2 = 0, \quad h_3^2 + h_4^2 = 0, \ldots\ldots h_{b-2}^2 + h_{b-1}^2 = 0,$$

which are respectively equivalent to

$$h_1 \pm \sqrt{-1}\,h_2 = 0, \quad h_3 \pm \sqrt{-1}\,h_4 = 0 \ldots h_{b-2} \pm \sqrt{-1}\,h_{b-1} = 0,$$

and if, taking the upper or lower sign as may be most convenient, we eliminate, by means of the last $\frac{1}{2}(b-1)$ equations $\frac{1}{2}(b-3)$ of the ratios $z_1 z_2, \ldots z_{b-1}$, we ultimately arrive at a simple equation which may be represented by

$$f\{\tfrac{1}{2}(b+1)\} = 0.$$

6. Next, let $f^3(10)=0$, then, a being the coefficient of z_1, add and subtract so as to put the last equation under the form

$$h_1^3 + z_1 f^2(9) + f^3(9) = 0 \ldots\ldots\ldots\ldots (11)$$

where $h_1 = z_1 + \dfrac{a}{3}$, and the two last functions are free from z_1. Then, if $f^2(9)=0$, (11) becomes

$$h_1^3 + f^3(9) = 0 \ldots\ldots\ldots\ldots (12)$$

* Those interested in this subject, are recommended to peruse the Mathematical Researches of Mr. Jerrard, where it is treated in a different manner, but with similar results.

But, as has been shown in the preceding paragraph, $f^2(9) = 0$ may, by giving a proper value to z_{10}, be decomposed into four simple equations. Thus we may change, by elimination, the last term of (12) into $f^3(4)$, for z_{10} has been already determined. Now $f^3(4)$ may be denoted by

$$p(z^3_2 + az^2_2 + \beta z_2 + \gamma),$$

and we are at liberty to suppose that $\beta = \frac{1}{3} a^2$, and $\gamma = \frac{1}{27} a^3$.

But these last equations are of the forms

$$f^2(3) = 0, \text{ and } f^3(3) = 0,$$

the former of which may, by paragraph 3, be reduced to $f(2) = 0$, and eliminating one of the ratios between this equation and $f^3(3) = 0$, (which has become $f^3(2) = 0$) we obtain $f^3(1) = 0$, a final cubic in z_3: and by means of these last operations $f^3(4)$ has been reduced to the form

$p(z_2 + \frac{a}{3})^3$ or h^3_2, hence $f^3(10) = 0$ may be put under the form

$$h^3_1 + h^3_2 = 0 \text{.........................} (13)$$

which is equivalent to three simple equations expressed by the formula $h_1 + (1)^{\frac{1}{3}} h_2 = 0$.

7. These results suffice for the present, but we shall afterwards see that our investigations may be carried still further, so as to enable us to arrive at the general theorem that "b may be assumed, so that $f^a(b) = 0$ can, by means of equations whose degrees do not exceed the a^{th}, be reduced to the form $\Sigma(h^a) = 0$, and from thence to the simple equation $f(c) = 0$, where c has an *indefinite* number of values," and, hence, that "any number of such functions can be simultaneously made equal to zero, without our having to solve an equation of a degree higher than the highest dimension of those functions."

8. By way of illustrating the application of these remarks to the transformation of algebraic equations, I shall give an outline of the method by which the general equation of the n^{th} degree may be deprived of its second, third, and fourth terms, reserving for another opportunity the discussion of particular cases. For this purpose (8) gives

$$2np_2 - (n-1)p^2_1 = 0 \text{....} (14) \qquad 6n^2p_3 - (n-1)(n-2)p^3_1 = 0 \text{..} (15)$$

To satisfy these equations, assume

$$v = \Lambda x^{\lambda} + \Lambda' x^{\lambda'} + \Delta'' x^{\lambda''} + \text{M} x^{\mu},$$

then p_1, p_2, and p_3 being symmetrical functions of v, (14) and (15) may be respectively denoted by

$$f^2(3) = 0 \text{......} (16) \qquad f^3(3) = 0 \text{..........} (17)$$

But we have before seen, that z_3 may be determined so as to reduce (16) and (17) respectively to $f(2) = 0$ and $f^3(2) = 0$, and eliminating z_2 between these latter equations, we have $f^3(1) = 0$, a final cubic in z_1, and z_2 is known from $f(2) = 0$.

9. Similarly, if it were required to destroy the second, third, fourth, and r^{th} terms, we should have to assume, for v, an expression consisting of 22 terms, and should then have, as before

$$f^2(21) = 0 \text{...} (18) \qquad f^3(21) = 0 \text{....} (19) \qquad f^r(21) = 0 \text{....} (20)$$

We then (see paragraph 4) decompose (18) into ten simple equations, by means of a quadratic in z_{21}. Thus (19) and (20) respectively become $f^3(10) = 0$ and $f^r(10) = 0$; and, by paragraph 6, these last may be

respectively reduced to $f(2)=0$ and $f^r(2)=0$; and eliminating z, between these two last equations, we have the final equation $f^r(1)=0$, an equation in x, of the r^{th} degree.

Again, let it be required to take away any given number of terms, and of the terms to be destroyed, let $p_r x^{n-r}$ be that which is lowest in dimension with respect to x, then, in general this may be done by means of equations whose dimensions do not exceed r. The limitations to this proposition will be considered hereafter.

The following remark will not be misplaced with reference to future applications of the above method. If $3ac=b^2$, and z be determined by the equation,

$$(9ad-bc)z+3bd-c^2=0, \text{ then}$$
$$x^{3n}+ax^3+bx^2+cx+d=0, \text{ may be depressed to}$$
$$(y+z)^n+Ay+B=0,$$

as may be shown by the same process as that referred to in the concluding sentence at page 84 of your last number. This is one of a class of equations which I shall at some time discuss with reference to the theory there made use of. It gives *solvible* forms of equations of the 5^{th}, 6^{th}, and other higher degrees. We may obtain others from the solution of a cubic noticed above. These will be mentioned when we come to inquire into the possibility of solving equations of the higher degrees.

Temple, 11*th March*, 1844.

Note.—The resolution of functions into the sums of powers will be found useful in other branches of science, besides the Theory of Equations, and, as will be shown, the operations in paragraph 3 and other similar ones are rendered simple by means of a law to which the quantities successively added are subject.

ON THE VARIATION OF PARAMETERS:

WITH REFERENCE TO LINES AND SURFACES OF THE FIRST AND SECOND ORDERS.

[*Mr. Rutherford.*]

Lines and surfaces of the first and second orders are expressed by equations involving several constant quantities, as well as the variables which are the co-ordinates of any point in the line or surface. By giving to the variables a series of values, we may trace, by a series of points determined from the proposed equation, the line or surface which that equation represents; but whilst all possible values are given to the variables, we may at the same time suppose any of the constants to be variable likewise, and thus there would result a series of lines or surfaces corresponding to the series of values given to the arbitrary constant. As this is a subject of considerable importance in the higher inquiries of Analytical Geometry, we shall illustrate it at some length, deferring its application for a future article.

I. *A line of the first order.*

The equation of a straight line, or a line of the first order, is

$$ay+bx+c=0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and when a, b, c are constant and given quantities, the straight line of which (1) is the equation, is completely determined in position; but if we su[illegible] that any one of the constants is variable as well as x and y, the e[illegible] (1) will still be the equation of a straight line, though undeter-

mined in position, on account of the arbitrary parameter, and therefore equation (1) must, in this case, represent a series of straight lines corresponding to the series of values which are given to the arbitrary constant.

(1) Let a be the variable quantity, and b and c constants; then if we make $y=0$, we shall have $bx+c=0$, and therefore $x=-\frac{c}{b}$; hence since the value of x is independent of the arbitrary constant, all the lines represented by (1) must, in this case, cut the axis of x in the same point, whose distance from the origin is $=-\frac{c}{b}$, whatever be the value of a; consequently, when a is supposed variable in equation (1), that equation expresses a series of straight lines, all passing through one and the same point in the axis of x, determined by making $y=0$ in the proposed equation.

(2) If b is the variable quantity; then if we make $x=0$, we get $ay+c=0$, or $y=-\frac{c}{a}$, a value independent of the variable b; and therefore, in this case, equation (1) expresses a series of straight lines all passing through the same point in the axis of y, determined in position from the equation $ay+c=0$.

(3) When c is the variable quantity, and a, b, constant; then dividing (1) by a, and transposing, we have

$$y=-\frac{b}{a}x-\frac{c}{a};$$

therefore, since a and b are constants, the tangent of the angle which the straight lines expressed by (1) make with the axis of x, is independent of the variable c, and is therefore the same, whatever may be the value of c; hence equation (1) expresses a series of straight lines, all parallel to that whose equation is $ay+bx=0$.

If when a is variable, b is zero, then the series of straight lines are all parallel to the axis of x; and when b is variable, whilst a is zero, the series of straight lines are all parallel to the axis of y.

II. *A line of the second order.*

The general equation of a line of the second order is

$$ay^2+bxy+cx^2+dy+ex+f=0 \ldots\ldots\ldots\ldots\ldots (2)$$

(1) If a is supposed variable, and all the other coefficients constant, then the locus of (2) is a series of curves which touch each other in the same two points, real or imaginary, in the axis of x.

This will be evident if we make $y=0$; for then we have the equation

$$cx^2+ex+f=0,$$

which has two roots, x_1 and x_2, either real or imaginary, and therefore the curves must all pass through those points in the axis of x, indicated by the values of x_1 and x_2, deduced from the last equation. To prove that the curves touch each other in those points, it will be necessary to show that they have a common tangent. Now the equation of a tangent to a line or curve of the second order, at the point $x'y'$, is

$$(2ay'+bx'+d)y+(2cx'+by'+e)x+dy'+ex'+2f=0,$$

and when the point is in the axis of x, we have $y'=0$; consequently, the last equation becomes

$$(bx'+d)y+(2cx'+e)x+ex'+2f=0,$$

which, by writing for x' the two values x_1 and x_2 gives either

$$(bx_1+d)y+(2cx_1+e)x+ex_1+2f=0,$$

$$\text{or, } (bx_2+d)y+(2cx_2+e)x+ex_2+2f=0.$$

But neither of these last equations contains the variable quantity a, and therefore they can only express the same two straight lines, whatever may be the value of a; consequently the curves touch each other in the two points in the axis of x, as above determined.

(2) When b is supposed variable, then equation (2) expresses a series of curves which intersect each other in the same *four* points, real or imaginary, of the co-ordinate axes.

For if we put successively $x=0$, and $y=0$, we have the two equations

$$ay^2+dy+f=0,$$

$$cx^2+ex+f=0,$$

both of which are independent of the variable b; and therefore the values of y and x obtained from these equations remain unchanged, whatever value may be assigned to b, and therefore, in this case, equation (2) expresses a series of curves intersecting each other in the same *four* points of the axes of co-ordinates, *two* of these points being in the axis of x and the other *two* in the axis of y; and these points are determined from the two preceding equations.

(3) Let c be the variable, then equation (2) expresses a series of curves which touch each other in the same two points, real or imaginary, of the axis of y.

This is proved in the same manner as the case of a being the variable.

(4) When d is supposed to vary, then equation (2) represents a series of similar and similarly situated curves, which cut each other in the same two points in the axis of x.

The curves are similar and similarly situated, because the criterion employed for ascertaining the nature of the curve involves simply the first three coefficients a, b, c, all of which are constant, and they cut the axis of x in the same two points, because when $y=0$, the coefficients of the resulting equation

$$cx^2+ex+f=0$$

are all constants.

(5) When e is the variable, then (2) expresses a series of similar and similarly situated curves, which cut each other in the same two points in the axis of y.

This is proved in a similar manner as the last case.

(6) When f is the variable quantity, then equation (2) expresses a series of similar and similarly situated concentric curves.

The curves are similar and similarly situated for the same reason as in case (4), and they are concentric, because the co-ordinates of the centre (α and β), viz.

$$\alpha=\frac{2ac-bd}{b^2-4ac}\,;\ \beta=\frac{2cd-be}{b^2-4ac}$$

ar[illegible]ident of the variable quantity.

III. *A surface of the first order.*

A plane is a surface of the first order, and its general equation is

$$az+by+cx+f=0 \qquad \text{.................. (3)}$$

(1) Let a be the variable; then if $z=0$, we have the equation

$$by+cx+f=0,$$

which is the equation of a straight line in the plane of xy, and is the same for all values of a; hence equation (3) represents a series of planes which cut the plane of xy in the same straight line determined by the equation $by+cx+f=0$.

(2, 3.) If either b or c be the variable; then it is shown in a similar manner that equation (3) represents a series of planes intersecting the planes of zx or zy in the same straight lines determined by the equations

$$az+cx+f=0.$$

$$\text{or } az+by+f=0.$$

(4.) Let f be the variable; then equation (3) expresses a series of parallel planes.

For the equations of its traces on the planes of zy and zx are respectively

$$az+by+f=0, \text{ or } z=-\frac{b}{a}y-\frac{f}{a},$$

$$az+cx+f=0, \text{ or } z=-\frac{c}{a}x-\frac{f}{a},$$

and therefore the angles which the traces make with the axes of y and x are constant, their trigonometrical tangents being expressed in terms of a, b, c; hence the series of planes, in this case, are all parallel to the plane which passes through the origin, whose equation is

$$az+by+cx=0.$$

IV. *A surface of the second order.*

The general equation of a surface of the second order is

$$az^2+a_1y^2+a_2x^2+bzy+b_1zx+b_2xy+cz+c_1y+c_2x+f=0 \qquad \text{.... (4)}$$

(1.) Let a be supposed variable, and all the other coefficients constant; then equation (4) expresses a series of surfaces which touch each other, and the curve of contact is a line of the second order in the plane of xy.

For if in (4) we make $z=0$, we shall have the equation

$$a_1y^2+a_2x^2+b_2xy+c_1y+c_2x+f=0 \qquad \text{................ (5)}$$

and as this equation expresses a line of the second order, and being independent of the variable parameter a, it is manifest that whatever value a may have, eq. (4) expresses a series of surfaces which cut the plane of xy in a line of the second order. To prove that the surfaces touch each other in this line of the second order, it will be necessary to show that they have a common tangent plane.

Let then $x'y'z'$, and $x''y''z''$ be the co-ordinates of any two points in the surface expressed by (4), then we have the equations

$$az'^2+a_1y'^2+a_2x'^2+bz'y'+b_1z'x'+b_2x'y'+cz'+c_1y'+c_2x'+f=0 \text{ .. (6)}$$

$$az''^2+a_1y''^2+a_2x''^2+bz''y''+b_1z''x''+b_2x''y''+cz''+c_1y''+c_2x''+f=0 \text{ .. (7)}$$

Subtracting the latter of those from the former, we have

$$a(z'+z'')(z'-z'')+a_1(y'+y'')(y'-y'')+a_2(x'+x'')(x'-x'')+b(z'y'-z''y'')$$
$$+b_1(z'x'-z''x'')+b_2(x'y'-x''y'')+c(z'-z'')+c_1(y'-y'')+c_2(x'-x'')=0 \text{... (8)}$$

But since $z'y'-z''y''=z'(y'-y'')+y''(z'-z'')$, *etc.*, the equation (8) becomes

$$\{a(z'+z'')+by''+b_1x''+c\}(z'-z'')+\{a_1(y'+y'')+bz'+b_2x'+c_1\}(y'-y'')$$
$$+\{a_2(x'+x'')+b_1z'+b_2y''+c_2\}(x'-x'')=0\ \ldots\ldots(9)$$

Now the equations of a straight line through the points $x'y'z'$ and $x''y''z''$ are

$$x'-x''=p\,(z'-z''),\ \text{and}\ y'-y''=q\,(z'-z'')\ldots\ldots\ldots\ldots(10)$$

where $$p=\frac{x-x'}{z-z'},\ \text{and}\ q=\frac{y-y'}{z-z'}.$$

Substituting equations (10) in (9) and rejecting the general factor $(z'-z'')$ we obtain the equation

$$a(z'+z'')+by''+b_1x''+c+\{a_1(y'+y'')+bz'+b_2x'+c_1\}q$$
$$+\{a_2(x'+x'')+b_1z'+b_2y''+c_2\}p=0\ldots(11)$$

Now let $z'=z''$, and consequently from (10) $x'=x''$ and $y'=y''$; then (11) becomes a tangent whose equation is

$$2az'+by'+b_1x'+c+(2a_1y'+bz'+b_2x'+c_1)q$$
$$+(2a_2x'+b_1z'+b_2y'+c_2)p=0\ \ldots\ldots\ldots\ (12)$$

Eliminating p and q by means of equations (10, 12) we get for the tangent plane at the point $x'y'z'$, the equation

$$(2az'+by'+b_1x'+c)\,(z-z')+(2a_1y'+bz'+b_2x'+c_1)(y-y')$$
$$+(2a_2x'+b_1z'+b_2y'+c_2)(x-x')=0\ldots(13)$$

Multiplying out, there results

$$(2az'+by'+b_1x'+c)z+(2a_1y'+bz'+b_2x'+c_1)y+(2a_2x'+b_1z'+b_2y'+c_2)x$$
$$-(2az'+by'+b_1x'+c)z'-(2a_1y'+bz'+b_2x'+c_1)y'$$
$$-(2a_2x'+b_1z'+b_2y'+c_2)x'=0\ \ldots\ldots(14)$$

But by equation (6) we have obviously

$$(2az'+by'+b_1x'+2c)z'+(2a_1y'+bz'+b_2x'+2c_1)y'$$
$$+(2a_2x'+b_1z'+b_2y'+2c_2)x'+2f=0,$$

and this being added to equation (14), reduces it to

$$(2az'+by'+b_1x'+c)z+(2a_1y'+bz'+b_2x'+c_1)y$$
$$+(2a_2x'+b_1z'+b_2y'+c_2)x+cz'+c_1y'+c_2x'+2f=0\ \ldots\ (15)$$

which is the equation of a plane tangent to the surface at the point $x'y'z'$.

Now, if in (15) we suppose the point to be in the plane xy, we have $z=0$, and the equation of the tangent plane becomes

$$(by'+b_1x'+c)z+(2a_1y'+b_2x'+c_1)y+(2a_2x'+b_2y'+c_2)x$$
$$+c_1y'+c_2x'+2f=0\ldots(16)$$

which is independent of the variable a, and hence whatever may be the value of the arbitrary constant, the surfaces must all touch each other in a line of the second order in the plane of xy.

(2, 3.) Let either a_1 or a_2 be the variable; then the locus of (4) is a series of surfaces which touch each other in a line of the second order, either in the plane of zx or zy, determined by making $y=0$, and $x=0$ successively in the proposed equation (4).

The reason of this is obvious from the preceding investigation.

(4.) Let b be the variable; then (4) denotes a series of surfaces which intersect each other in the same *six* points, real or imaginary, of the axes of co[illegible]ates.

[illegible]de

For if we put $z=0$ and $y=0$; then $a_2x^2+c_2x+f=0$

$$z=0 \text{ and } x=0 \text{ —— } a_1y^2+c_1y+f=0$$

$$x=0 \text{ and } y=0 \text{ —— } az^2+cz+f=0.$$

Now each of these three equations is independent of the arbitrary constant; and therefore whatever value may be assigned to b, the equation (4) denotes a series of surfaces which cut the axes of co-ordinates in the same *six* points, two in each axis.

(5, 6.) Let either b_1 or b_2 be the variable; then the last three equations are still independent of either the one or the other of these variables, and consequently the series of surfaces, in either case, cut the axes of co-ordinates in the same six points as when b is the variable.

(7.) When c is taken as the variable; then (4) expresses a series of similar and similarly situated surfaces of the second order which intersect each other in the same line of the second order in the plane of xy.

The surfaces are manifestly similar and similarly situated, because the first six coefficients, on which the similarity depends, are constant, and that they cut each other in the same line of the second order in the plane of xy, is shown exactly as in the first case.

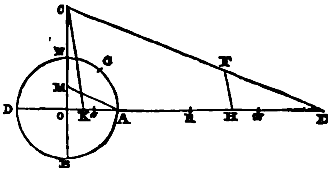
(8, 9.) The cases when c_1 and c_2 are made the variables, give similar results.

(10.) Lastly, let f be the variable; then (4) expresses a series of similar and similarly situated concentric surfaces. For the co-ordinates of the centre are independent of the value of f, and therefore, whether the surfaces have or have not centres, they are nevertheless concentric, and they are similar and similarly situated for the reason assigned in the preceding cases.

APPROXIMATE RECTIFICATION OF THE CIRCLE.

[*Mr. Hugh Godfray, Jersey.*]

The following approximation to the rectification of the circle is closer, I believe, than any one yet given. It is true to seven decimal places, while Pioché's method (which is the most exact of those collected by Mr. Davies*) is true only to six. My construction, also, is about one-fourth shorter than his, as I have tested on several occasions, by comparing the times in which the same person, equally acquainted with both methods, could perform them. Still, it is not so much for its practical utility as for the remarkable extent of approximation, that this construction will, probably, be interesting: for it is such, that in a circle whose radius is equal to that of the earth, the error in the circumference would only be about 18 inches.



Let ABDN be a circle, whose centre is O; draw any diameter D A, and produce it indefinitely; and draw BOC perpendicular to DA cutting the circle in B and N: then make OC equal to the diameter, and AE equal to double the diameter. Join CE, and on it set off EF equal to the diameter.

Again, with centre A and distance OA describe an arc cutting the circumference in G; set off DG from E to H;

* Leybourn's *Math. Repos.* Vol. vi. p. 158; or Hutton's *Course*, 12th edit. Vol. i. p. 400.

join FH; draw CK parallel to FH; and make Ka equal OK. Also draw AM parallel to EC; and set off AR = AN, and Rω = BM. Then will $a\omega$ be equal to the semi-circumference, very nearly.

$$\begin{aligned}\text{For } a\omega &= O\omega - Oa = O\omega - 2OK = OA + AR + R\omega - 2(OE - EK)\\ &= 1 + AN + BM - 2(5 - EK) = 1 + \sqrt{2} + 1 + OM - 10 + 2EK\\ &= \sqrt{2} - 8 + OM + 2EK = \sqrt{2} - 8 + \frac{OA.OC}{OE} + 2.\,\frac{EH.EC}{EF}\\ &= \sqrt{2} - 8 + \frac{1.2}{5} + 2\,\frac{DG\sqrt{(EO^2 + OC^2)}}{2} = \sqrt{2} - 7\tfrac{3}{5} + \sqrt{3}.\sqrt{25+4}\\ &= 1{\cdot}41421356237 + 9{\cdot}32737905309 - 7{\cdot}6 = 3{\cdot}14159261546.\end{aligned}$$

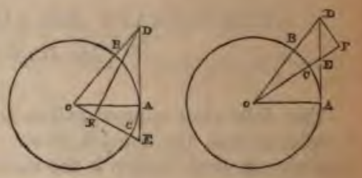
Jersey, Feb. 16, 1844.

Mr. Godfray also favoured us with two other constructions, the operations of which are very brief and simple: but, as might be expected, their degree of approximation is much less than some known constructions furnish.

FUNDAMENTAL FUNCTIONS OF TWO ARCS.

[*From a Correspondent.*]

At p. 85 of the *Mathematician*, is an attempt of mine to obtain these formulas from considerations suggested by the method by which Lagrange deduced the fundamental formulæ of Spherical Trigonometry. I now offer another mode of investigation, better adapted to the purposes of systematic instruction; and which, probably, will be found more simple in its character than any one yet before the public.



From the junction A of the arcs AB, AC draw the radius OA; also the tangent at A meeting OB, OC in D and E, and the perpendicular DF from D upon OE. Also denote the radius OA by unity, and the arcs AB, AC by a and b. Then, in the first figure BC $= a + b$; and in the second, BC $= a - b$.

Now the triangles EOA, EDF are, in both figures similar, and hence DF : DE : : OA : OE, or OE.DF = DE.OA; or since DF = OD sin DOE,

$$\sin DOE = \frac{DA \pm AE}{OD.OE}; \text{ or again}$$

$$\sin(a \pm b) = \frac{\tan a \pm \tan b}{\sec a \sec b} = \sin a \cos b \pm \cos a \sin b.$$

Again, in the same figures,

$$DE^2 = OD^2 + OE^2 - 2OE.OD \cos DOE, \text{ and}$$

$$DE^2 = AD^2 + AE^2 \pm 2AD.AE \text{ (\textit{Euc.} ii., 4 and 7.)}$$

And by subtraction and *Euc.* i., 47, we have

$$2 \mp 2 \tan a \tan b = 2 \sec a \sec b \cos(a \pm b),$$

$$\text{or, } \cos(a \pm b) = \frac{1 \mp \tan a \tan b}{\sec a \sec b} = \cos a \cos b \mp \sin a \sin b.$$

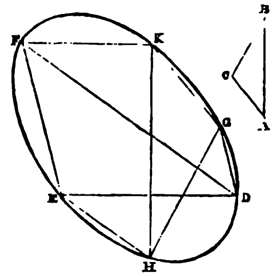
[illegible]*ast, April* 8, 1844. Y.

PROPOSITIONS ON THE CONIC SECTIONS.

[*Mr. James Dalmahoy, Edinburgh.*]

PROP. I. PROBLEM.

To inscribe in a conic section, given in position, a triangle whose sides shall be parallel to three given lines.



Let DHEFKG be a conic section given in position, in which it is required to inscribe a triangle whose sides shall be parallel to those of the given triangle ABC.

Find D, E, F points in the curve, such that tangents applied to them, shall be parallel respectively to the sides AB, AC, and BC of the given triangle ABC. Join D, E, F; and through D draw the chord DG parallel to EF; through E draw E H parallel to D F; and through F draw FK parallel to D E. The triangle formed by joining the points G, H, K is similar to the given triangle ABC.

For* the chord HK is parallel to the tangent at D, and is therefore parallel to the side AB of the given triangle. In like manner GK is proved to be parallel to AC, and GH to BC. Consequently the triangle GHK, inscribed in the conic section, having its sides HK, KG and GH respectively parallel to the sides AB, AC and BC of the given triangle ABC, the two triangles are similar.

Cor.—An infinite number of triangles may be described in any conic section which shall be similar to a given triangle.

Scholium.—It is evident that the given triangle must be drawn so that none of its sides may be parallel to the transverse axis of the hyperbola or the axis of the parabola.

PROP. II. PORISM.

The Porism which, according to Playfair,† was the last but one in the third book of Euclid's Porisms, relates to the circle only. The object of the following demonstration is to shew that if the expression, "equal segments," be substituted for "equal circumferences," in the enunciation of that porism, it becomes applicable to all the conic sections.

LEMMA.

If in a conic section two chords cut off equal segments, the straight lines joining the corresponding extremities of these chords are parallel.

Let ADA and BCB be equal segments of a conic section cut off by the chords AD and BC; the line AB is parallel to DC. (The reader will easily sketch the figure.)

For if not, then from B, the extremity of AB, draw the chord BE parallel to DC, and join DE. Since the chord BE is parallel to CD, the lines BC and ED, joining their extremities, cut off‡ equal segments BCB and EDE.

* Simson's Conic Sections, B. IV., Prop. xxix., Cor. 2.

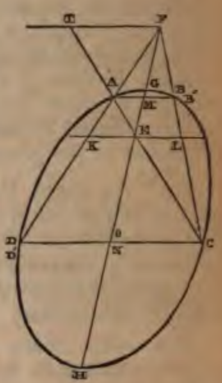
† On the Origin and Investigation of Porisms, (§ 10,) Playfair's Works, Vol. iii., p. 200.

‡ Stone's Translation of de l'Hôspital's Conic Sections, B. v., P. 3., and Cor. 2.

But, by hypothesis, BCB is equal to ADA, consequently EDE is equal to ADA, or the part is equal to the whole, which is absurd; AB is therefore parallel to DC.

THE PORISM.

A point being given, either without or within a conic section given in position, if there be drawn anyhow through that point a line cutting the curve in two points; another point may be found in the same diameter, or its extension, in which the given point lies, such, that if two lines be drawn from it to the points, in which the line already drawn cuts the curve, these two lines will cut off from the conic section equal segments.



Let ABCD be a given conic section; E a given point without or within it; AC a chord drawn anyhow through E. Another point F may be found in the same diameter GH, or its extension, in which E lies, such, that if the chord AD be drawn through A and F, and the chord BC through C and F, they will cut off equal segments ADA and BCB.

Analysis.—Join A, B and C, D, cutting the diameter GH in M and N. Through E draw a line parallel to AB and intersecting the lines FA and FC, produced if necessary, in K, L; also through F draw a parallel to the same line AB, intersecting AC, or its extension, in T.

The segments ADA and BCB being, by hypothesis, equal, the line AB is, by the foregoing lemma, parallel to DC. Also, it may easily be proved that F, the intersection of the lines AD and BC joining the extremities of the parallel chords AB and BC, lies in the diameter to which these chords are conjugate. But, by hypothesis, the points F and E lie in the diameter GH, consequently the chords AB and DC are conjugate to GH, and they are divided equally in M and N by that diameter.

If the point F be supposed to coincide with the centre O in the ellipse and hyperbola, it can no longer be proved, by the foregoing process of reasoning, that AB and DC are conjugate to GH, for then more than one diameter can be drawn through F without coinciding with each other. It may, however, be shewn that, in such case, the porism becomes altogether indeterminate in the ellipse, and impossible in the hyperbola; for in the former curve, if a line be drawn from any point whatever to cut the curve in two points, chords, that is diameters, drawn through F to these points, divide the ellipse into two equal parts: and in the hyperbola a diameter meets the opposite branches each in one point only, and therefore cannot cut off a segment. It might further be easily proved, that when F coincides with the centre, the point E must be infinitely distant from it, and consequently is not, in strictness, a given point.

It hence follows that, in all possible cases of the porism, the diameter GH divides equally the chords AB and DC in M and N. It also divides into equal parts, KE and EL, the line KL, which is parallel to AB and is intercepted between the diverging lines FA and FC or their extension. Furth[illegible]ce the extremities of the parallel lines TF and EL, TF and

KE, are connected by diverging lines intersecting in C and A, it follows that

CE : CT : : EL : TF; and AE : AT : : KE : TF;

but EL = KE; and hence by identity of ratios,

CE : CT : : AE : AT,

that is, the chord AB is divided harmonically at the point E and by a line FT conjugate to the diameter GH at T. Consequently the line FT is the polar to the pole E, and is given in position*: also F, its intersection with the diameter GH, is given.

Composition.—In the diameter GH, or its extension, in which the given point E lies, find, in the ellipse and hyperbola, a point F, such that OF may be a third proportional to OE and OG; and in the parabola such that GF may be equal to GE: through F draw FT conjugate to the diameter GH; the line FT is the polar† to E the given point. Through E draw any chord AC meeting FT in T. Through the point F draw to the extremities of AC, the chords AD and BC; and through the points A, C and E draw lines each parallel to TF; the line through A meeting CB, or its extension, in B′; the line through C meeting AD, or its extension, in D′; and the line through E meeting AD and BC, or their extensions, in K, L.

The segments ADA and BCB, cut off from the conic section by the chords AD and BC drawn through the points F and A; and F and C are equal.

For, from the properties of diverging lines cut by parallels,

EL : TF : : CE : CT; and KE : TF : : AE : AT.

Again the chord AC is harmonically ‡ divided in E and T; hence

CE : CT : : AE : AT; and EL : TF : : KE : TF;

and hence again, EL = KE. Wherefore it readily follows that AM=MB′ and CN=ND′. Therefore MB′ and ND′ are ordinates to the diameter GH, and B′ coincides with B, and D′ with D; consequently AB and DC are parallel chords, and hence the segments ADA and BCB, cut off by chords drawn through F, A and F, C, are equal.

* Davies' Hutton, Vol. ii., Cor. to Prop. iv.

† Davies' Hutton, Vol. ii., P. 176, Prop. ii.

‡ Stone's Translation of de l'Hôspital's Conic Sections, B. v., P. 3., also Cor. 2.

ON THE THEORY AND APPLICATION OF POLES AND POLARS.

[*Mr. Fenwick.*]

It is universally admitted that the French mathematicians were the inventors of this powerful agent of discovery and investigation. Probably its origin is to be traced to the mighty mind of Monge. Gergonne and Poncelet, his pupils, listened to the effusions of his transcendent genius and received on their minds the spark that was to surround them with an unextinguishable halo of glory. To which of these two geometers the honour of the invention is to be assigned is uncertain. They were both claimants, and the dispute that arose between them in connexion with their claim has been recorded but not decided. With respect to the name pole and polar, Chasles, in his "Apercu Historique," writes, that Servois was the first

that gave the appellation of *Pole* to a point analogous to that which forms the object of our present consideration, and that Gergonne subsequently introduced the application of the correlative term *Polar*.

The terms *pole* and *polar*, in reference to the theory of poles and polars, would seem scarcely known to English mathematicians, if we were to form an opinion from books published in this country. The *germs* only of the theory are given in one or two works. This slowness to adopt the improvements of continental mathematicians is not easily accounted for, and it is the more remarkable in the present instance, as the Polar theory has been the parent of so much valuable mathematical knowledge. This theory may be characterised as a new power, — a new engine, which would, however, have fallen short of half its greatness, had it not been brought into the domain of the Geometry of Descartes. In co-operation with the co-ordinate theory, it has brought to light numerous treasures that lay buried in the mine. Improvement in its mechanism, simplification in its details, and good order and arrangement in its processes, have gradually augmented its efficacy and value. The most complicated relations between points and lines of the second order; between planes and surfaces of the second degree, are established without difficulty by a theory so peculiarly calculated to exhibit them in their various combinations.

We propose in the following pages to commence with an entirely elementary introduction of the theory, and from a desire to render the subject easy of comprehension to students in general, we will address ourselves, not to the accomplished mathematician, who needs not our assistance, but to the learner. We may therefore be pardoned, if from the simple particular cases we proceed step by step to the general theorems. In this respect we shall but imitate our great instructer Euclid. As the theory is peculiarly applicable to the conic sections or lines of the second degree, this branch will receive considerable attention and amplification. It will be seen with what facility, as compared with the ordinary methods, we are able to deduce the equations of some of the principal lines to those curves. We especially allude to the equations of the diameter and tangent. This application of the polar theory seems new, and is capable, no doubt, of considerable extension. Our mode of theoretical elucidation is also new, as far as we are able to ascertain, and it appears to be much more simple than any we have hitherto had an opportunity of examining.

We proceed, then, to a developement of the theory in reference to a point and line in plano.

Definition.—Two systems of points which are so related, that to each *point* of the one system there is a corresponding *line* of the other system, and to each point of the second system a corresponding line of the first system, are named a *reciprocal system*. The point and the line are named in reference to each other *pole* and *polar*, or *reciprocal polars*.

Or the two systems are so connected that the co-ordinates of a given point in the one, and the variable co-ordinates of a line in the other, are *interchangeable* in the equation which expresses their relation.

1. *To find the equation of reciprocity between a point and its polar.*

Let x_1y_1 be the co-ordinates of a point in the one system, and x_2y_2 those of a point in the other, referred to any axes whatever.*

* In this paper both systems are supposed to be referred to the *same axes*, but a similar mode of investigation is applicable when the axes are different.

Then $$y - y_1 = m(x - x_1) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
and $$y - y_2 = m_1(x - x_2) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$
will be the equations of *any* two lines through the respective points, m and m_1 being *arbitrary.*

Combine (1) and (2) by addition, and we have for a line through the intersection of (1) and (2), the equation

$$2y = (m + m_1)x + (y_1 + y_2) - (mx_1 + m_1x_2) \quad\ldots\ldots\ldots\ldots(3)$$

Let (3) be identical with the equation

$$y = ax + b \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and we get from (3, 4)

$$m + m_1 = 2a, \text{ and } y_1 + y_2 - (mx_1 + m_1x_2) = 2b.$$

Eliminating m_1 from these, there results the relation

$$2(ax_2 + b) - (y_1 + y_2) + m(x_1 - x_2) = 0 \quad\ldots\ldots\ldots\ldots(5)$$

Hence, in consequence of the indeterminateness of m, we get finally from (5)

$$2(ax_2 + b) - (y_1 + y_2) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$
and $$x_1 - x_2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

If, then, x_1y_1 are given, x_2y_2 are also given by (6, 7); and if x_2y_2 are given, x_1y_1 are also given by the same equations. There is, therefore, no relation of pole and polar in this case, inasmuch as to a given point in the one system there is not a corresponding line in the other.

Again, let

$$(y - y_1) - A(x - x_1) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(8)$$
and $$(y - y_1) - B(x - x_1) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

be the equations of *any* two lines which pass through the point (x_1y_1). Combine (8, 9) by multiplication, and put $A + B = p$, $AB = q$; then we have for a curve of the second degree through the point (x_1y_1) the equation

$$y^2 - pxy + qx^2 - (2y_1 - px_1)y - (2qx_1 - py_1)x$$
$$+ y_1^2 + qx_1^2 - px_1y_1 = 0 \ldots(10)$$

Similarly, the equation of a curve of the second degree through the point (x_2y_2) is

$$y^2 - p_1xy + q_1x^2 - (2y_2 - p_1x_2)y - (2q_1x_2 - p_1y_2)x$$
$$+ y_2^2 + q_1x_2^2 - p_1x_2y_2 = 0 \ldots(11)$$

Combine (10, 11) by addition, and we have

$$2y^2 - (p + p_1)xy + (q + q_1)x^2 - (2y_1 - px_1 + 2y_2 - p_1x_2)y$$
$$-(2qx_1 - py_1 + 2q_1x_2 - p_1y_2)x + y_1^2 + y_2^2 + qx_1^2 + q_1x_2^2 - px_1y_1 - p_1x_2y_2 = 0$$
$$\ldots(12)$$

Let the last equation be *identical* with the equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \quad\ldots\ldots\ldots\ldots(13)$$

Hence from (12, 13) we get the following equations

$$a(p + p_1) = -2b; \quad a(2y_1 - px_1 + 2y_2 - p_1x_2) = -2d;$$
$$a(q + q_1) = 2c; \quad a(2qx_1 - py_1 + 2q_1x_2 - p_1y_2) = -2e;$$
$$a(y_1^2 + y_2^2 + qx_1^2 + q_1x_2^2 - px_1y_1 - p_1x_2y_2) = 2f.$$

Eliminating p, p_1, q and q_1, from these, we have for the polar of the point (x_1y_1), the equation

$$(2ay_1 + bx_1 + d)y_2 + (2cx_1 + by_1 + e)x_2 + dy_1 + ex_1 + 2f = 0;$$

or for the polar of the point (x_2y_2), the equation

$$(2ay_2+bx_2+d)y_1+(2cx_2+by_2+e)x_1+dy_2+ex_2+2f=0.$$

Writing x and y for the variables in these equations, they become

$$(2ay_1+bx_1+d)y+(2cx_1+by_1+e)x+dy_1+ex_1+2f=0\,..(14)$$

and $(2ay_2+bx_2+d)y+(2cx_2+by_2+e)x+dy_2+ex_2+2f=0\,..(15)$*

2. *General equation of reciprocity between a point and its polar.*

If tu be the co-ordinates of a point in the one system, and xy those of a point in the other; then generally the equation

$$(au+bt+c)y+(a_1u+b_1t+c_1)x+a_2u+b_2t+c_2=0\,........(16)$$

or $$(ay+a_1x+a_2)u+(by+b_1x+b_2)t+cy+c_1x+c_2=0\,........(17)$$

will express the polar of the point (xy) if x and y are regarded as constant, or it will express the polar of (tu) if these co-ordinates are considered constant.

3. *Co-ordinates of the pole.*

The co-ordinates (x_1y_1) of the pole of a given polar line can be found by considering this polar line and (14) or (16), as the case may be, *identical*, and equating the co-efficients of the like powers of the variables.

4. *The polar of the point of intersection of two lines passes through the poles of these lines: and conversely, the polars of two points intersect in the pole of the line joining these points.*

Let x_1y_1, x_2y_2, be the co-ordinates of the two points, and t_1u_1 those of the point of intersection. Then the polars of the points t_1u_1; x_1y_1; x_2y_2; are obviously expressed by the equations

$$(au_1+bt_1+c)y+(a_1u_1+b_1t_1+c_1)x+(a_2u_1+b_2t_1+c_2)=0\,......(18)$$

$$(ay_1+a_1x_1+a_2)u+(by_1+b_1x_1+b_2)t+(cy_1+c_1x_1+c_2)=0\,......(19)$$

$$(ay_2+a_1x_2+a_2)u+(by_2+b_1x_2+b_2)t+(cy_2+c_1x_2+c_2)=0\,......(20)$$

And since (19, 20) intersect in (t_1u_1), these equations must be satisfied by the co-ordinates t_1u_1, so that they become

$$(ay_1+a_1x_1+a_2)u_1+(by_1+b_1x_1+b_2)t_1+(cy_1+c_1x_1+c_2)=0,$$

and $$(ay_2+a_1x_2+a_2)u_1+(by_2+b_1x_2+b_2)t_1+(cy_2+c_1x_2+c_2)=0;$$

or arranging according to y_1 and x_1, *etc.*,

$$(au_1+bt_1+c)y_1+(a_1u_1+b_1t_1+c_1)x_1+(a_2u_1+b_2t_1+c_2)=0,$$

and $$(au_1+bt_1+c)y_2+(a_1u_1+b_1t_1+c_1)x_2+(a_2u_1+b_2t_1+c_2)=0.$$

Now the two last equations are precisely what (18) becomes, by substituting successively for x and y the values x_1y_1, and x_2y_2. The equation (18), then, is satisfied by the co-ordinates x_1y_1 and x_2y_2, and therefore the polar of the point (t_1u_1) passes through the points x_1y_1 and x_2y_2.

5. The following general theorem is readily deduced from the last art., viz.

If a straight line turns round a fixed point, its pole describes a straight line which is the polar of the fixed point: and conversely, if a point moves in a straight line, its polar turns round a fixed point, the pole of that straight line.

For it readily follows from art. 4, that the poles of three or more straight lines which intersect in the same point, are in the same straight line, which

* This mode of establishing the polar theory differs from any other that I have seen. It is based on this principle,—that since the co-ordinates of the point and line are to be *interchangeable* in the equation which expresses their relation, these co-ordinates must be *equally involved in the several equations* that are employed for determining the final relation.

is the polar of the point of intersection; and conversely, if three or more points be in the same straight line, then the polars of these points intersect in a point which is the pole of that line.

6. *The poles of all straight lines that are parallel to the same straight line are in one and the same straight line.*

This is deduced from art. 4, by considering that three or more parallel straight lines can be supposed to intersect in a point infinitely distant.

This brings us to the subject of *diameters* (see Mathematician, No. I., p. 38). We shall in some of the subsequent articles enter a little more fully into this subject.

7. *To find in general the diameter of a system.*

Let $$y = px + q \qquad\qquad (21)$$
be the equation of a line in which p is *constant* and q *variable*. Then (21) will represent all lines that are parallel to the same straight line; and the *locus of the poles* of all the lines which are parallel to (21) will be found (arts. 3 and 6) from the supposed *identity* of (16) and (21). Hence we have the equation
$$a_1u + b_1t + c_1 + p\,(au + bt + c) = 0 \qquad\qquad (22)$$
which is a diameter of the system (tu), and is conjugate to the line (21).

8. *Centre of a system.*

All the diameters of a system intersect in the same point *(which is named the centre of the system)*, or are parallel to one another.

For let p be *arbitrary* in (22), and this line will then represent all the diameters in the system (tu). Hence, in consequence of the arbitrary quantity p, we have
$$a_1u + b_1t + c_1 = 0, \text{ and } au + bt + c = 0 \qquad\qquad (23)$$
The values of u and t, *the co-ordinates of the centre of the system*, derived from (23), are real and finite, unless $\frac{b}{a} = \frac{b_1}{a_1}$, inasmuch as the denominator of u or t is $\frac{b_1}{a_1} - \frac{b}{a}$.

9. *To find the equation of a diameter to a line of the second degree.*

Let $$y = px + q \qquad\qquad (24)$$
be the equation of a line in which p is constant, and q variable; then equating the coefficient of x in this line with that of x in (14), we get for a diameter, the equation
$$2cx_1 + by_1 + e + p(2ay_1 + bx_1 + d) = 0,$$
or writing y, x for y_1, x_1, and arranging for y and x, we have
$$(2ap + b)y + (bp + 2c)x + (dp + e) = 0 \qquad\qquad (25)$$
which is the equation of a diameter to a curve of the second degree represented by (13) art. 1.

10. *Criterion of Conjugate Diameters.*

If the equation to any diameter of a line of the second degree, represented by (13) art. 1, be
$$y = px + q;$$
the equation of the diameter conjugate to it will be
$$y = -\frac{bp + 2c}{2ap + b}\,x + q'.$$

This criterion may also be expressed by the condition

$$2app' + b(p + p') + 2c = 0,$$

where p' is the coefficient of x in the second equation.

For the equation of the diameter conjugate to the line whose equation

$$y = px + q,$$

is, by (25) art. 9,

$$(2ap+b)y+(bp+2c)x+(dp+e)=0.$$

Hence, in order that this line may be parallel to that whose equation is

$$y = p'x + q',$$

we must have

$$\frac{bp+2c}{2ap+b} = -p', \text{ or } p' = -\frac{bp+2c}{2ap+b} \text{..............} (26)$$

and therefore the equation $y = p'x + q'$, becomes

$$y = -\frac{bp+2c}{2ap+b}x + q'.$$

This establishes the first criterion.

Again, the expression (26) readily reduces to

$$2app' + b(p + p') + 2c = 0,$$

which is the second.

11. *Centre of a conic section.*

The equation of a diameter (art. 9) being

$$(2ap+b)y+(bp+2c)x+(dp+e)=0,$$

$$\text{or} \quad (2ay + bx + d)p+(by + 2cx+e)=0;$$

hence, if p be arbitrary, we have

$$2ay+bx+d=0, \text{ and } by+2cx+e=0,$$

from which the co-ordinates of the centre are found in the usual way. The expression $\frac{b_1}{a_1} - \frac{b}{a}$, found in art. 8, becomes in this case, $\frac{b}{2a} - \frac{2c}{b}$. Hence the co-ordinates of the centre are real and finite, unless $b^2 - 4ac = 0$.

12. *To find the equation of a tangent to a line of the second degree.*

Let x_1y_1 be the point of contact; then the tangent in general is

$$y - y_1 = A(x - x_1) \text{...................} (\alpha)$$

Now this line (α) is parallel to the line which is conjugate to the diameter passing through x_1y_1, that is to the line (14); for (x_1y_1) is any point in the diameter. Hence (α) is parallel to the line whose equation is

$$(2ay_1+bx_1+d)y+(2cx_1+by_1+e)x+dy_1+ex_1+2f=0;$$

and therefore we have

$$A = -\frac{2cx_1 + by_1 + e}{2ay_1 + bx_1 + d}.$$

Substituting this in (α) and reducing, we get

$$(2ay_1+bx_1+d)y+(2cx_1+by_1+e)x-(2ay^2_1+bx_1y_1+dy_1)$$
$$-(2cx^2_1+bx_1y_1+ex_1)=0 \text{....} (\beta)$$

And again, since x_1y_1 is a point in the curve, we have

$$ay^2_1+bx_1y_1+cx^2_1+dy_1+ex_1+f = 0,$$

$$\text{or } -2(ay^2_1+cx^2_1+bx_1y_1) = 2(dy_1+ex_1+f).$$

Hence (β) becomes

$$(2ay_1+bx_1+d)\,y+(2cx_1+by_1+e)x+dy_1+ex_1+2f=0^*\text{......}(27)$$

which is the equation of a tangent at the point x_1y_1.

13. It follows from the last art. that

The tangent at any point (x_1y_1) of a conic section can be regarded as the polar of this point; and conversely, this point can be regarded as the pole of the tangent.

For the condition of reciprocity between the point (x_1y_1) and the line (27) is obviously fulfilled, inasmuch as the co-ordinates are *interchangeable.*

14. Hence also the following theorem:

The polar of a point (x_1y_1), in reference to a line of the second degree, is parallel to the diameter which is conjugate to that diameter which passes through the point (x_1y_1).

15. The following theorems are also readily deduced from articles 4, 5, and 13:

(*a*) *The line joining the points of contact of two tangents to the same conic section is the polar of the point of intersection of these tangents. Or the polar in respect of a point without a conic section is the chord of contact of the two tangents drawn from that point.*

(*b*) *The point of intersection of two chords of a conic section is the pole of the line joining the points of intersection of both pairs of tangents at the extremities of these chords.*

(*c*) *If the angular point of an angle moves upon a straight line, whilst the lines containing the angle touch a curve of the second degree, whereby the angle is in general varied, the chords of contact turn about the same point, the pole of the line upon which the angular point moves. And conversely, if the chord of a curve of the second degree turns about a fixed point, the tangents at the extremities of the chords intersect in the same straight line, the polar of the fixed point.*

16. *Let a polygon of any number of sides be inscribed in a conic section, and let tangents be drawn through its angular points, so that a polygon of the same number of sides may circumscribe the same conic section: then the two polygons are reciprocal polars of one another.*

For, since (art. 13) "the tangent at any point of a conic section can be regarded as the polar of that point," and also (art. 15) "that the line joining the points of contact of two tangents to the same conic section is the polar of the point of intersection of the tangents;" it follows, that the sides of the inscribed polygon are the polars of the angular points of the circumscribed one; and conversely, the sides of the circumscribed polygon are the polars of the angular points of the inscribed one.

17. Let ABCDEF and *abcdef*, be the inscribed and circumscribed hexagons to a conic section, of which *ab* passes through A, *bc* through B, *etc.* Produce AF, CD to meet in K; BC, FE to meet in H; and AB, ED to meet in G.

* Independently of the great facility of the polar method in finding the equation of a diameter, tangent, *etc.*, it enables us also to find the equation of a tangent without employing the indeterminate expression $\frac{0}{0}$.

Then the polars of the points K, H and G, are the lines which join a, d; c, f; and b, e; respectively, art. 15, (b). Hence if we denote H, K and G by (0 0); (x_1y_1); and (x_2y_2); the polars of these points will be denoted by the following equations, art. 1, (14),

$$(cf) \ldots\ldots\ldots\ldots dy+ex+2f=0 \ldots\ldots\ldots\ldots (28)$$

$$(ad) \ldots (2ay+bx+d)y_1+(2cx+by+e)x_1+dy+ex+2f=0 \ldots (29)$$

$$(be) \ldots (2ay+bx+d)y_2+(2cx+by+e)x_2+dy+ex+2f=0 \ldots (30)$$

Take (28) from (29) and also from (30), and we have the equations

$$(2ay+bx+d)y_1+(2cx+by+e)x_1=0 \ldots\ldots\ldots\ldots (31)$$

$$(2ay+bx+d)y_2+(2cx+by+e)x_2=0 \ldots\ldots\ldots\ldots (32)$$

Again, combine (31) and (32), and we get

$$(2ay+bx+d)\left(\frac{y_1}{x_1}-\frac{y_2}{x_2}\right)=0.$$

Hence $\quad 2ay+bx+d=0$, and $y_1x_2-y_2x_1=0$.

The latter of which shows that K, H, G are in the same straight line. We have therefore the following theorem:

If the three pairs of opposite sides of a hexagon inscribed in a conic section be produced to meet, the three points of intersection will range in the same straight line.—(Pascal's Hexagram.)

And again (art. 5), "if three or more points be in the same straight line, then the polars of these points intersect in a point which is the pole of that line." It will be obvious then, that since the points K, H, G, are in the same straight line, the polars cf; ad; be; intersect in the same point. Hence also the following theorem:

If a hexagon be circumscribed to a conic section, the three diagonals which join the three pairs of opposite summits, will pass through the same point. (Brianchon's theorem.)

18. *The polar of the focus of a conic section is the directrix.*

Let us take the parabola.

The equation of the parabola when referred to the axis and tangent at the vertex, as axes of co-ordinates, is

$$y^2=4mx.$$

Comparing this with the general equation

$$ay^2+bxy+cx^2+dy+ex+f=0,$$

we find

$$a=-\frac{e}{4m};\ b=0;\ c=0;\ d=0;\ f=0.$$

Hence (14) art. 1, or the polar of the point x_1y_1 becomes

$$-y_1y+2m(x+x_1)=0.$$

But if the focus be the pole, we have $x_1=m$, $y_1=0$, and therefore the polar of the focus is

$x+m=0$, which is the equation of the directrix.

A similar property is easily shown to belong to the ellipse and hyperbola.

19. *If a chord be drawn through the focus of a conic section, and tangents be drawn to the extremities of the focal chord, these tangents will intersect in the directrix.*

This theorem follows at once from the last art. and one of the properties deduced in art. 15, viz.

"If the chord of a conic section turns about a fixed point, the tangents at the extremities of the chord intersect in the same straight line, the polar of the fixed point."

20. *If a quadrilateral be inscribed within a conic section, and from its angular points, as points of contact, a quadrilateral be described about the conic section: then we have the following theorems,*

(*A*) *If the four pairs of opposite sides be produced to meet, the four points of intersection will be situated in one line.*

(*B*) *The four diagonals of these two quadrilaterals will pass through the same point.*

(*C*) *The point and line will be a pole and polar in reference to the conic section.*

Let ABCD be the inscribed quadrilateral (the student will readily sketch the fig.), and *a b c d* the circumscribed one; *ab*, *bc*, etc. passing through A, B, etc. Let DA, CB meet in M; CD, BA in L; *da*, *cb* in N; and *cd*, *ba* in P. Now the polars of L, N, and P are, by (*a*) and (*b*), art. 15, the lines *bd*, BD, AC. Hence, denoting the points L, N and P by x_1y_1, x_2y_2, and x_3y_3, we get by (14), art. 1, the following equations:

$$(bd)\ldots.(2ay+bx+d)y_1+(2cx+by+e)x_1+dy+ex+2f=0\ldots.(33)$$

$$(\mathrm{BD})\ldots.(2ay+bx+d)y_2+(2cx+by+e)x_2+dy+ex+2f=0\ldots.(34)$$

$$(\mathrm{AC})\ldots.(2ay+bx+d)y_3+(2cx+by+e)x_3+dy+ex+2f=0\ldots.(35)$$

Take (33) from (34), and also from (35), and we have

$$(2ay+bx+d)(y_1-y_2)+(2cx+by+e)(x_1-x_2)=0\ldots\ldots(36)$$

$$(2ay+bx+d)(y_1-y_3)+(2cx+by+e)(x_1-x_3)=0\ldots\ldots(37)$$

These are the equations of two lines through the intersections of (33, 34), and (33, 35).

Again, combine (36) and (37), and we get

$$(2ay+bx+d)\left\{\frac{y_1-y_2}{x_1-x_2}-\frac{y_1-y_3}{x_1-x_3}\right\}=0.$$

Hence we have

$$\frac{y_1-y_2}{x_1-x_2}-\frac{y_1-y_3}{x_1-x_3}=0, \text{ or } (x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)=0.$$

Now this is the relation amongst the co-ordinates of three points in the same straight line. It follows, then, that L, P, N, are in the same straight line.

In a similar way it may be shewn that M, P, N, are in the same straight line; and therefore the four points L, M, N, P, are in a straight line.

This establishes theorem (*A*).

Again, when three or more points are in the same straight line, their polars intersect in the same point, the pole of that line (art. 5); and since the lines AC, BD, *ac*, *bd*, are the polars of the points P, N, M, L, these lines will intersect in the same point, the pole of the line LMNP. This establishes theorems (*B*) and (*C*).

21. *If the pairs of opposite sides* DA, BC, *and* AB, DC, *of a quadrilateral* ABCD, *inscribed in a conic section, be produced to meet in* M

and L, *and if the diagonals meet in* Q; *then* ML *is the polar of the pole* Q, MQ *the polar of the pole* L, *and* QL *the polar of the pole* M.

For take LBA and LCD as axes of co-ordinates, and put LB $= a$; LA $= a_1$; LC $= \beta$; LD $= \beta_1$. Then the following equations will be obvious:

$$(\text{DA})\ldots\ldots\frac{y}{\beta_1}+\frac{x}{a_1}=1\ldots\ldots(38)$$

$$(\text{BC})\ldots\ldots\frac{y}{\beta}+\frac{x}{a}=1\ldots\ldots(39)$$

$$(\text{AC})\ldots\ldots\frac{y}{\beta}+\frac{x}{a_1}=1\ldots\ldots(40)$$

$$(\text{BD})\ldots\ldots\frac{y}{\beta_1}+\frac{x}{a}=1\ldots\ldots(41)$$

Combine (38, 39), and also (40, 41), by subtraction, and we get for ML and QL, the equations

$$(\text{ML})\ldots\ldots\left(\frac{1}{\beta_1}-\frac{1}{\beta}\right)y+\left(\frac{1}{a_1}-\frac{1}{a}\right)x=0\ldots\ldots\ldots(42)$$

$$(\text{QL})\ldots\ldots\left(\frac{1}{\beta_1}-\frac{1}{\beta}\right)y-\left(\frac{1}{a_1}-\frac{1}{a}\right)x=0\ldots\ldots\ldots(43)$$

Now by (C) art. 20, ML is the polar of the pole Q; hence if we denote Q by (x_1y_1), then by (14) art. 1, (42) becomes

$$(2ay_1+bx_1+d)y+(2cx_1+by_1+e)x=0;$$

and therefore by (43) the equation of LQ is

$$(2ay_1+bx_1+d)y-(2cx_1+by_1+e)x\ldots\ldots\ldots\ldots\ldots\ldots(44)$$

Hence since x_1y_1 are the co-ordinates of Q, and xy those of M, it follows that (44) or the line LQ is the polar of the point M. Again, by art 4, the poles of two straight lines which intersect are in the polar of the point of intersection; hence MQ is the polar of the pole L. The theorem is consequently established.

22. If a quadrilateral be inscribed in a conic section, and another circumscribed about the same conic section, as in art. 20, then

The diagonals of the circumscribed quadrilateral will pass through the points in which the opposite sides of the inscribed one intersect.

For, the figure being the same as in arts. 20, 21, *ac* is the polar of M by art. 20, and QL is the polar of M by art. 21; hence since *ac* passes through the point Q, it must tend to the point L. In a similar way it may be shewn that *bd* tends to the point M. Hence the truth of the theorem.

Scholium 1.—*Harmonic pole and polar.*

If from a point in the plane of a curve of the second degree a line be drawn to meet the curve and the polar of the given point; this line will be divided harmonically at the four points. This property has led to the names, *harmonic pole and polar*. As we have not used this property in the preceding investigations, we have not given a demonstration of it.*

Scholium 2.—The investigations contained in these pages will, we trust, suffice for illustrating the theory of poles and polars. We would remind the student that the theorem established in art. 1, subject to slight modification, is equally applicable to *any* of the conic sections. We have given an

* The property alluded to in this scholium is very effective in the *geometrical* mode of [illegible]shing the polar theory. In *Davies's Edition of Hutton's Course*, vol. ii. p. 174, [illegible]ent will find many of the properties of poles and polars deduced by the *geometrical*

example illustrative of this in art. 18, and we shall add another in the next art.

23. *If from a point* P *in an ellipse perpendiculars* PG, PH, *be let fall on two equal conjugate diameters; then the diagonal of the parallelogram upon* PG, PH, *is normal to the ellipse at the point* P.

Let O be the centre. Take OG, OH as axes of co-ordinates, and denote the point P by x_1y_1. Then θ being the angle of ordination, we have

$$\mathrm{OG} = x_1 + y_1\cos\theta, \text{ and } \mathrm{OH} = y_1 + x\cos\theta.$$

Hence the co-ordinates of the middle point (M) of GH are

$$\tfrac{1}{2}(x_1 + y_1\cos\theta) \text{ and } \tfrac{1}{2}(y_1 + x_1\cos\theta);$$

and therefore the equation of PM is

$$y - y_1 = \frac{y_1 - x_1\cos\theta}{x_1 - y_1\cos\theta}(x - x_1)\ldots\ldots\ldots\ldots\ldots(45)$$

Again, the normal at the point P is evidently at right angles to the polar of that point. Now the equation of an ellipse, when referred to equal conjugate diameters as axes of co-ordinates, is

$$y^2 + x^2 - a^2 = 0.$$

Comparing this with the general equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

we get $\quad b = 0, \quad c = a, \quad d = 0, \quad e = 0,$ and $f = -a^2$.

These values reduce (14), art. 1, to

$$y_1y + x_1x - a^2 = 0, \text{ or } y = -\frac{x_1}{y_1}x + \frac{a^2}{y_1}\ldots\ldots\ldots\ldots\ldots(46)$$

which is the polar of the point P.

The perpendicular from P (x_1y_1) on the line (46) is

$$y - y_1 = -\left\{\frac{1 - \frac{x_1}{y_1}\cos\theta}{-\frac{x_1}{y_1} + \cos\theta}\right\}(x - x_1) = \frac{y_1 - x_1\cos\theta}{x_1 - y_1\cos\theta}(x - x_1).$$

Hence, since the last equation is identical with (45), the proposition is established.

24. *A triangle* ABC *inscribed in a conic section is such that the line bisecting one of its interior angles* (A) *is a normal to the curve; then the side* (BC) *opposite to the bisected angle passes through the pole of the normal.—(Annales des Mathematiques.)*

Before we enter upon an investigation of this problem we shall first find the co-ordinates of the pole (art. 3) of a polar supposed to coincide with the axis of x.

Now, x_1y_1 being the pole, the polar in reference to the conic section $ay^2+bxy+cx^2+dy+ex+f=0$, is (art. 1.)

$$(2ay_1+bx_1+d)y+(2cx_1+by_1+e)x+dy_1+ex_1+2f=0.$$

Hence in order that this line may coincide or be identical with the axis of x, we must have

$$2cx_1+by_1+e=0, \text{ and } dy_1+ex_1+2f=0.$$

From these we get

$$x_1 = \frac{2bf - de}{2dc - eb} \text{ and } y_1 = \frac{e^2 - 4cf}{2dc - eb}\ldots\ldots\ldots\ldots\ldots(\gamma)$$

Let now the conic section be referred to the normal and tangent as axes of co-ordinates; then its equation will be

$$ay^2+bxy+cx^2+ex=0 \dots\dots\dots\dots\dots\dots (47)$$

Hence, since in this case, $d=0$, and $f=0$, the co-ordinates (γ) of the pole of the normal become

$$x_1=0, \text{ and } y_1=-\frac{e}{b}.$$

Again, let the equations of the sides AC, AB, of the inscribed triangle, be

$$y+\alpha x=0, \text{ and } y-\alpha x=0.$$

Combining these by multiplication, we get for a line of the second degree through the points A, B, C, the equation

$$y^2-\alpha^2x^2=0.$$

Let this be combined with (47) so that the resulting equation may be that of a straight line: this will evidently be the line BC. Hence we have for BC the equation

$$by+(c+a\alpha^2)x+e=0.$$

Now for $x=0$, $y=-\frac{e}{b}$, and as these are equal to the co-ordinates of the pole of the normal (found above) the enunciated property is consequently established.

HORNER ON ALGEBRAIC TRANSFORMATION.

[*Continued from page* 112.]

The theorem just obtained is manifestly of extreme generality of application, and we proceed to illustrate some of the applications of its prolific principles.

15. Respecting the kinds of functions to which this process is applicable, the obvious general conclusion from the conditions of the theorem is, that it applies to all such expressions as may be conceived to have been originally formed by such a process. For example, it naturally adapts itself to formulæ arranged according to the powers of an unknown quantity, because such a formula may be supposed to have been self-derived in indefinite succession by means of the divisor $\frac{x-r_0}{x_0}=\frac{x}{x}$, or which amounts to the same thing, by means of a series of multipliers originally equal to zeros, and kept in that state by the admission of increments all equal to zero.

16. To functions of Differences it is not so obviously suitable; since the series of multipliers by which connected sets of differences must be supposed to be connected, is uniformly 1, 1, 1, *etc.*, which could not happen if the constant increment 1 of the variable were introduced according to our law of transformation.

It will, however, be seen that differences enter into our continuous formulæ in the divided form $\Delta, \frac{1}{1.2}\Delta^2, \frac{1}{1.2.3}\Delta^3, \dots \frac{1}{1.2..n}\Delta^n$; against which no objection lies. For in this case, the connecting multipliers are $n, n-1$, ... 2, 1; which is in strict accordance with the rule.

In fact, the adaptation of the rule to any function, in which either the connecting multipliers or the factorial terms compose an arithmetical series, is conspicuous.

17. In accordance with this connection we may also remark:

(1) That no one increment can be employed in more than n successive transformations of a formula of n dimensions. Its influence is withdrawn from the terms one by one in retrogradation; so that after m transformations since its introduction, it no longer affects any of the last m terms. Whence, it likewise follows:—

(2) That no multiplier can be composed of more increments than are denoted by the proper index of the term which is multiplied by it.

18. A similar property, of course, belongs to the varieties of x. Each term will contain just as many varieties as are denoted by the proper index (m); and these will always be the last m varieties which were introduced: so that no individual factor can remain in any of the last $m+1$ terms during more than m transformations after its admission; nor in any term, after n transformations.

19. And hence, we readily collect that if the same factor x_p, and consequently the same increment 0, be used m times successively, the last $m+1$ terms will contain only x_p and its powers, and the last m coefficients will be the same as they would be if the like system were pursued during n transformations, in which case the formula would become a pure function of x_p. No further change would then be effected by persisting in this plan,—a fact which evinces that the assumption made in a former article (15) was not altogether hypothetical.

20. The solution of equations by means of these *pure transformées*, the only class in fact which mathematicians have hitherto been accustomed to regard as accessory to that object, has been largely explained in my former paper. I revert to the subject in passing, just to shew that its principle is included in that of the present theorem.

If r be constant during n transformations, or which is the same, if n successive increments be 0, the general theorem in art. 13, assumes the form

$$
\begin{array}{llllllll}
A_0 & B_0 & C_0 & \ldots\ldots & K_0 & L_0 & M_0 & N_0 \\
0 & A_1 r & B_1 r & \ldots\ldots & H_1 r & K_1 r & L_1 r & M_1 r \\
\hline
A_1 & B_1 & C_1 & \ldots\ldots & K_1 & L_1 & M_1 & N_1 \\
0 & A_2 r & B_2 r & \ldots\ldots & H_2 r & K_2 r & L_2 r & \\
\cline{1-7}
A_2 & B_2 & C_2 & \ldots\ldots & K_2 & L_2 & M_2 & \\
0 & A_3 r & B_3 r & \ldots\ldots & H_3 r & K_3 r & & \\
\cline{1-6}
A_3 & B_3 & C_3 & \ldots\ldots & K_3 & L_3 & & \\
 & \&c. & & \ldots\ldots & & \&c. & &
\end{array}
$$

and the final sums N_1, M_2, L_3, ... will be the coefficients of formula 1, (art. 2.)

21. This process represents, in a different notation the identical mode of transformation which is given in *art.* 14 of my former paper: and the improved method given in *art.* 16, and more explicitly in *art.* 18, of the same paper, is readily deduced from it. For, since $B_m = B_{m-1} + A_m r$, and so

of the rest, if we multiply the equation by r, and make $B_m r = {}_mB$, we have ${}_mB = {}_{m-1}B + {}_mAr$. The addends, therefore, after the first course, may be derived each from the nearest addends above and on the left hand of it, instead of from the sum beneath the latter; and by a mental process when r is a digit: a method which requires only half the space, and half the number of multiplicand figures which are employed in that of the preceding article.* The result is

A_0	B_0	C_0	....	L_0	M_0	N_0
0	$A_1r = {}_1A$	$B_1r = {}_1B$	....	$K_1r = {}_1K$	$L_1r = {}_1L$	$M_1r = {}_1M$
A_1	B_1	C_1	....	L_1	M_1	N_1
	${}_1A + Ar = {}_2A$	${}_1B + {}_2Ar = {}_2B$	....	${}_1K + {}_2Hr = {}_2K$	${}_1L + {}_2Kr = {}_2L$	
	${}_2A + Ar = {}_3A$	${}_2B + {}_3Ar = {}_3B$	....	${}_2K + {}_3Hr = {}_3K$	M_2	
	....	....	....	L_3		
	to $= {}_nA$	to $= {}_{n-1}B$	....			
A_n	B_{n-1}	C_{n-2}	....			

Cor.—Hence $B_m = {}_mA + \frac{{}_{m-1}B}{r}$, *etc.*, or the amount of each column is simply equal to the sum of the lowest term in it added to $\frac{1}{r}$th of the lowest in the next.

If after these n transformations, we exchange r for $r + r_1$, and consequently the increment 0 for r_1, the succeeding course of transformations will begin according to the general law of *art.* 14, precisely as they begin at the foot of the General Synopsis in the former essay.

22. The last theorem, with a slight modification, furnishes perhaps the easiest and most elegant method possible of obtaining the suite of Differences appertaining to every coefficient of a given formula. For, when $r = 1$, it is obvious that the sum of all the addends to any term of the given formula is equal to the difference of that term; or that ${}_1M = \Delta N_0$, ${}_1L + {}_2L = \Delta M_0$, *etc.*, as in the following practical model:—

A_0	B_0	C_0	....	L_0	M_0	N_0
	${}_1A$	${}_1B$	....	${}_1K$	${}_1L$	${}_1M$
	${}_2A$	${}_2B$	....	${}_2K$	${}_2L$	
	${}_3A$	${}_3B$	....	${}_3K$		
	:	:				
	ΔB_0	ΔC_0		ΔL_0	ΔM_0	ΔN_0

where we perceive that the first course of addends are the successive sums

* The importance of the process here referred to by Mr. Horner has not attracted the slightest attention from mathematicians. Its being a little more complex in the *investigation*, led me to prefer the other, in a hasty and imperfect sketch of Horner's method, which I drew up for an *elementary* work, in 1835: but for a more general and enlarged treatise on numerical solution, long reflection has led me to consider that the method of this article would

of the given coefficients, omitting the last; and each of the following courses consists of the successive sums of the next preceding course, stopping short of the last term.

By subjecting these differences to a similar operation, but of course, one dimension lower, we shall obviously find *their* differences; and from these second differences, the third; and so on, in order. The work will necessarily terminate when its object is effected, viz. at the n^{th} course; all the dimensions being then exhausted.

Ex. (Lagrange, Eq. N., p. 16.) Find the differences due to the coefficients of $x^3-63x+189=0$.

	1	0	-63	$+189$
		1	1	-62
		1	2	
		1		
1st diffs.		3	3	-62
			3	6
			3	
2nd diffs.			6	6
3rd diffs.				6

23. Lagrange employs the differences of the absolute term in aid of the usual process for investigating the *limits* of the roots of an equation; as answering, with less labour, the purpose of substituting 0, 1, 2, *etc.*, for x. Their utility in this respect is, however, superseded by the facility and elegance which our general theorem accommodates itself to this particular case; viz. the case of a constant increment, 1.

When $x_m = x - m$, or $r_m = m$, the increment of r (*art.* 14) is constantly 1, and the theorem assumes the following form:

A_0	$B_0 \ldots\ldots\ldots K_0$	L_0	M_0	N_0
0	$A_1 \ldots\ldots\ldots H_1$	K_1	L_1	M_1
A_1	$B_1 \ldots\ldots\ldots K_1$	L_1	M_1	N_1
0	$2A_2 \ldots\ldots\ldots 2H_2$	$2K_2$	$2L_2$	M_2
A_2	$B_2 \ldots\ldots\ldots K_2$	L_2	M_2	N_2
0	$3A_3 \ldots\ldots\ldots 3H_3$	$3K_3$	$2L_3$	M_3
A_3	$B_3 \ldots\ldots\ldots K_3$	L_3	M_3	N_3
0	$4A_4 \ldots\ldots\ldots 4H_4$	$3K_4$	$2L_4$	M_4
A_4	$B_4 \ldots\ldots\ldots K_4$	L_4	M_4	N_4
	&c. &c.		&c.	

Here the *multiplier* for each term is invariably *the smaller of two*

be the most suitable. That sketch, however, was so fortunate as to attract attention to the method: and the attempt which I then made has tended to render Horner's researches and methods more generally known and consequently better appreciated.

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exponents: viz. that of the power to which it belongs and that oj transformée in which it stands. Consequently, as might indeec directly inferred from what is said in *art.* 17, the multipliers in the n^{th} following transformations are constantly n, $n-1$, $n-2$, ... 3, 2, 1

For convenience in practice, the indices and exponents in question be numbered as is done in the annexed example; where m^p directs to tiply p times by m, after which the multiplier is continually diminished

Ex. Find the limits of the positive roots of $x^3-63x+189=0$.

	1	0	—63	189
	0	1	1	—62
1^3....	1	1	—62	127
	0	2	6	—56
2^2....	1	3	—56	71
	0	3	12	—44
3^1....	1	6	—44	27
	0	3	18	—26
	1	9	—26	1
	0	3	24	—2
	1	12	—2	—1
	0	3	30	28
	1	15	28	27
		&c.	&c.	&c.

The absolute terms of these transformations being equal to the resı substituting 0, 1, 2, ... for x in the given equation (*Cor. art.* 12, changes of signs which they exhibit between the 4th and 5th and the 5t 6th transformations, shew that one of the roots begins with 4 and the with 5.*

24. If at any point in a series of operations such as this, it sho

* When greater rapidity is admissible, the marginal exponents may be varied acco For example, if we begin by substituting a instead of 1 for x, the continued mult during n transformations will not be succeeded by $m-1$ but by $m-a$; after wl continued deduction of 1 takes place. This abrupt change may be indicated by mp , When $m-a$ becomes equal to n, the constant indication n^1 takes place as usual. Tl be substituted in the first intance in the above example, we have

	1	0	—63	189
	0	3	9	—162
3^3........	1	3	—54	27
	0	4	28	—26
4^2, 1....	1	7	—26	1
	0	5	24	—2
5^1, 2.....	1	12	—2	—1
	0	3	30	28
3^1........	1	15	28	27.

desirable to superadd to the indications of the absolute term, those of the signs, *etc.* of the *pure* transformée, we need only introduce the increment 0 n times successively; that is, observe the set of multipliers employed in finding the term in question, and *continually rejecting one from the left hand, transform* with the rest. When the work is exhausted, the set of final sums will constitute the pure transformée required.

Though not absolutely necessary, it is always convenient, to apply this transformation at those points from which the ulterior solution is to proceed; for instance, to the fourth and fifth transformées in the last example. In each of these cases the multipliers were 3, 2, 1.* Using, therefore, first 2, 1, and then 1 only, we have

$$\begin{array}{rrrr|rrrr} & 1 & 9 & -26 \quad 1 & & 1 & 12 & -2 \quad -1 \\ 2^1 \ldots\ldots & 1 & 11 & -15 & 2^1 \ldots\ldots & 1 & 14 & 12 \\ 1 \ldots\ldots & 1 & 12 & & 1 \ldots\ldots & 1 & 15 & \\ & 1 & & & & 1 & & \end{array}$$

The pure transformées are therefore

$$x_4^3 + 12x_4^2 - 15x_4 + 1, \text{ and } x_5^3 + 15x_5^2 + 12x_5 - 1\,;$$

which may be solved by any of the usual methods.

25. Our general formulæ present another variation of the method of limits, which, adhering to pure transformées, keeps every requisite indication constantly in view; but wants the neatness and rapidity of the former in ordinary cases, and particularly when applied to high powers.

In *art.* 21, make $r=1$, and we have this theorem:

$$\begin{array}{lllllll} & A_0 & B_0 & C_0 \ldots\ldots\ldots K_0 & L_0 & M_0 & N_0 \\ \hline 1^n \ldots\ldots & A_1 & B_1 & C_1 \ldots\ldots\ldots K_1 & L_1 & M_1 & N_1 \\ \hline 1^{n-1} \ldots & A_2 & B_2 & C_2 \ldots\ldots\ldots K_2 & L_2 & M_2 & \\ \hline 1^{n-2} \ldots & A_3 & B_3 & C_3 \ldots\ldots\ldots K_3 & L_3 & & \\ \hline \&c. & \&c. & & \&c. & \&c. & & \end{array}$$

where each course of coefficients is formed of the successive sums of the preceding course, stopping short of the final term except at the first summation. The new transformée is

$$A_{n+1} \quad B_n \quad C_{n-1} \ldots\ldots\ldots K_4 \quad L_3 \quad M_2 \quad N_1\,;$$

and the entire process consists of a series of similar transformations.

26. This is the method of *Budan*; from which the following, derived from *art.* 22, differs only in a slight accession of convenience.

$$\begin{array}{lllllll} & A_0 & B_0 & C_0 \ldots\ldots\ldots K_0 & L_0 & M_0 & N_0 \\ 1^{n-1} & & {}_1A & {}_1B \ldots\ldots\ldots {}_1H & {}_1K & {}_1L & {}_1M \\ 1^{n-2} & & {}_2A & {}_2B \ldots\ldots\ldots {}_2H & {}_2K & {}_2L & \\ 1^{n-3} & & {}_3A & {}_3B \ldots\ldots\ldots {}_3H & {}_3K & & \\ & & \text{to} & \ldots\ldots\ldots {}_4H & & & \\ & \ldots\ldots & \ldots\ldots & \ldots\ldots & & & \\ & & {}_nA \quad {}_{n-1}B & & & & \\ \hline & & A_{n+1} & B_n \ldots\ldots\ldots K_4 & L_3 & M_2 & N_1 \end{array}$$

* In this example as discussed in the last note, the transformées in question are the 2nd and 3rd; and the multipliers are 4, 4, 1, and 5, 2, 1. T. S. D.

where the courses of addends are the successive sums of the previous course, always omitting the last term; and the amount of any column is found or verified by simply adding the lowest term in it to the lowest in the next column.—(*Cor. art.* 21.)

Either of these processes, when combined with the *discriminative* use of *Budan's** valuable criterion of impossible roots, and that of De Gua exemplified in my last paper, forms a very complete instrument of initial solution. (See *Leybourn's Repository*, No. 18.)

* I cheerfully retract the opinion I formerly held respecting this author's mode of operating, which appears to be as here stated, and not as I understood him, by figurate multipliers. As *Legendre* seems to have fallen into the same error as myself, I may be allowed to ascribe the circumstance fully as much to the defect of clearness in the writer's statement of his principle, as to the rapidity with which I skimmed over his work, satisfied with finding that he had not anticipated my discovery.

(To be continued.)

THE EQUATION TO THE PARABOLA, REFERRED TO TWO TANGENTS AS AXES.

[*Mr. J. W. Elliott, Greatham, Stockton-on-Tees.*]

The equation of a curve of the second degree, referred to any axes of co-ordinates, is

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0 \ldots\ldots\ldots\ldots (1)$$

in which, let x and y be zero successively: then

$$Ay^2 + Dy + F = 0 \ldots\ldots (2) \quad | \quad Cx^2 + Ex + F = 0 \ldots\ldots (3)$$

The values of x and y in these equations either give the points of contact, or of intersection of the curve with the axes: but as the curve is referred to two tangents, they give the points of contact; and consequently the two values of x must be equal, as also those of y. Let them be

$$x = a, \; y = b \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4, 5)$$

and then by the theory of equations, we have from these, and (2, 3)

$$\frac{D}{A} = -2b, \; \frac{F}{A} = b^2; \; \frac{E}{C} = -2a, \; \frac{F}{C} = a^2.$$

Whence

$$A = \frac{F}{b^2}, \text{ and } D = -2Ab = -\frac{2F}{b},$$

$$C = \frac{F}{a^2}, \text{ and } E = -2Ca = -\frac{2F}{a}.$$

Now that the locus of (1) should be a parabola, it must be subjected to the condition

$$B^2 - 4AC = 0, \text{ or } B = \pm 2\sqrt{AC} = \pm \frac{2F}{ab}.$$

Substituting these values of A, B, C, D, E in (1), dividing by F, and reducing, we obtain

$$a^2y^2 \pm 2abxy + b^2x^2 - 2a^2by - 2ab^2x + a^2b^2 = 0 \ldots\ldots\ldots\ldots (6)$$

which represents either a parabola or one of its varieties; and as the equation

is affected with a double sign, it will be necessary to examine it under both aspects. Taking first the positive sign, (6) may be written in the form

$$\{(ay+bx)-ab\}^2=0; \text{ therefore}$$

$$ay+bx=ab; \text{ or } \frac{y}{b}+\frac{x}{a}=1$$

an equation to a straight line passing through the points $a0$, $0b$; which, in this case, is the line drawn through the points of contact of the axes with the curve. Taking now the negative sign, (6) may be written, after adding $4abxy$ to each side, in the form

$$\{(ay+bx)-ab\}^2=4abxy.$$

Extracting, and we have

$$ay+bx-ab=\pm 2\sqrt{abxy}, \text{ or}$$

$$(\sqrt{ay}+\sqrt{bx})^2=ab; \text{ therefore}$$

$$\sqrt{ay}+\sqrt{by}=\sqrt{ab}, \text{ or } \frac{\sqrt{y}}{\sqrt{b}}-\frac{\sqrt{x}}{\sqrt{a}}=1,$$

which is the equation to the parabola: it bears an analogy to the symmetrical equations of the straight line, circle (under certain conditions), ellipse, and hyperbola.

When $a=b$: then $\sqrt{y}+\sqrt{x}=\sqrt{a}$, and the locus is still a parabola. For an examination of this particular case, the student is referred to page 65 of Waud's very elegant and instructive treatise on Algebraic Geometry.

May 3rd, 1844.

ON THE FUNCTION $\Gamma(x+1)$.

[*Mr. Weddle, Newcastle on Tyne.*]

$$\text{Let } S_m=1^{-m}+2^{-m}+3^{-m}+4^{-m}+\ldots\ldots$$

$$s_m=1^{-m}-2^{-m}+3^{-m}-4^{-m}+\ldots\ldots$$

$$\sigma_m=1^{-m}+3^{-m}+5^{-m}+7^{-m}+\ldots\ldots$$

Then it may easily be shown that

$$s_m=\left(1-\frac{1}{2^{m-1}}\right)S_m, \text{ and } \sigma_m=\left(1-\frac{1}{2^m}\right)S_m \ldots. (1)$$

In *De Morgan's Diff. and Int. Calculus*, pp. 577—580, it is shown that if $\Gamma(x+1)=\int_0^\infty e^{-v}.v^x.dv$, then will

$$\Gamma(x+1)=x\Gamma(x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\Gamma(x).\Gamma(1-x)=\frac{\pi}{\sin\pi x}, \text{ or } \Gamma(1+x).\Gamma(1-x)=\frac{\pi x}{\sin\pi x}\ldots\ldots\ldots(3)$$

and $\log\Gamma(x+1)=-\gamma x+\frac{1}{2}S_2x^2-\frac{1}{3}S_3x^3+\frac{1}{4}S_4x^4-\ldots\ldots\ldots\ldots(4)$
where $\gamma=\cdot5772157\ldots.=$ value of $1^{-1}+2^{-1}+3^{-1}\ldots.+n^{-1}-\log n$, when $n=\infty$.

Find the value of $\log\Gamma(x+1)+\log\Gamma(1-x)$, from (4), and compare it with (3),

$$\therefore \log\frac{\pi x}{\sin\pi x}=S_2x^2+\tfrac{1}{2}S_4x^4+\tfrac{1}{3}S_6x^6+\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Hence (4) reduces to

$$\log\Gamma(x+1)=\tfrac{1}{2}\log\frac{\pi x}{\sin\pi x}-\gamma x-\tfrac{1}{3}S_3x^3-\tfrac{1}{5}S_5x^5\ldots\ldots\ldots(6)$$

In (5) change x into $\frac{1}{2}x$, and deduct (5) from the result. Hence by (1)

$$\log\cos\tfrac{1}{2}\pi x=-\sigma_2x^2-\tfrac{1}{2}\sigma_4x^4-\tfrac{1}{3}\sigma_6x^6-\ldots\ldots\ldots\ldots\ldots(7)$$

In like manner, from (1) and (5),

$$\log\frac{\pi x}{\sin\pi x}-2\log\frac{\frac{1}{2}\pi x}{\sin\frac{1}{2}\pi x},\text{ that is, }\log\frac{\tan\frac{1}{2}\pi x}{\frac{1}{2}\pi x}=s_2x^2+\tfrac{1}{2}s_4x^4+\tfrac{1}{3}s_6x^6+\ldots(8)$$

Bearing (1) in mind, we get from (4)

$$\log\Gamma(x+1)-\log\Gamma\left(\frac{x}{2}+1\right)=-\tfrac{1}{2}\gamma x+\tfrac{1}{2}\sigma_2x^2-\tfrac{1}{3}\sigma_3x^3+\tfrac{1}{4}\sigma_4x^4-\ldots(9)$$

$$\text{and }\log\Gamma(x+1)-2\log\Gamma\left(\frac{x}{2}+1\right)=\tfrac{1}{2}s_2x^2-\tfrac{1}{3}s_3x^3+\tfrac{1}{4}s_4x^4-\tfrac{1}{5}s_5x^5-\ldots(10)$$

Now (7) and (9) give

$$\log\Gamma\left(\frac{x}{2}+1\right)-\log\Gamma(x+1)-\tfrac{1}{2}\log\cos\tfrac{1}{2}\pi x=\tfrac{1}{2}\gamma x+\tfrac{1}{3}\sigma_3x^3+\tfrac{1}{5}\sigma_5x^5+\ldots(11)$$

Also from (8) and (10)

$$2\log\Gamma\left(\frac{x}{2}+1\right)-\log(x+1)+\tfrac{1}{2}\log\frac{\tan\frac{1}{2}\pi x}{\frac{1}{2}\pi x}=\tfrac{1}{3}s_3x^3+\tfrac{1}{5}s_5x^5+\tfrac{1}{7}s_7x^7+\ldots(12)$$

Let $Sn^{-m}=1^{-m}+2^{-m}+3^{-m}\ldots\ldots\ldots+n^{-m}$

and $P_n=(1+x)(1+\frac{1}{2}x)(1+\frac{1}{3}x)\ldots\left(1+\frac{1}{n}x\right)$

$$\therefore\log P_n=\log(1+x)+\log(1+\tfrac{1}{2}x)+\log(1+\tfrac{1}{3}x)\ldots+\log\left(1+\frac{1}{n}x\right)$$

$$=Sn^{-1}.\,x-\tfrac{1}{2}Sn^{-2}.\,x^2+\tfrac{1}{3}Sn^{-3}.\,x^3-\tfrac{1}{4}Sn^{-4}.\,x^4+\ldots\ldots$$

Deduct $\log n^x=x\log n$;

$$\therefore\log\frac{P_n}{n^x}=(Sn^{-1}-\log n)\,x-\tfrac{1}{2}Sn^{-2}.\,x^2+\tfrac{1}{3}Sn^{-3}.\,x^3-\tfrac{1}{4}Sn^{-4}.\,x^4+\ldots$$

Now, by (4), the limit (as n increases) of the right hand member is $-\log\Gamma(x+1)$ $\therefore\log\Gamma(x+1)=$ limit of $-\log\dfrac{P_n}{n^x}=$ limit of $\log\dfrac{n^x}{P_n}$.

$$\text{Hence, }\Gamma(x+1)=\text{limit of }\frac{n^x}{P_n}=\text{limit of }\frac{n^x}{(1+x)(1+\frac{1}{2}x)(1+\frac{1}{3}x)\ldots(1+\frac{1}{n}x)}\quad(13)$$

Since, limit of $\left(\dfrac{n}{n+1}\right)^x$ is 1, the preceding may be written

$$\Gamma(x+1)=\text{limit of }\frac{(n+1)^x}{(1+x)(1+\frac{1}{2}x)(1+\frac{1}{3}x)\ldots(1+\frac{1}{n}x)}.$$

But $(n+1)^x=\dfrac{2^x}{1^x}\cdot\dfrac{3^x}{2^x}\cdot\dfrac{4^x}{3^x}\ldots\ldots\dfrac{(n+1)^x}{n^x}$; hence by substitution and making n infinite,

$$\Gamma(x+1)=\frac{1^{-x}.2^x}{1+x}\cdot\frac{2^{-x}.3^x}{1+\frac{1}{2}x}\cdot\frac{3^{-x}4^x}{1+\frac{1}{3}x}\ldots\ldots\ldots\ldots(14)$$

by (14), $\Gamma(x+1).\,\Gamma(1-x)=\dfrac{1}{(1-x^2)(1-\frac{x^2}{4})(1-\frac{x^2}{9})\ldots\ldots}$

$\therefore$ by (3), $\sin\pi x=\pi x\,(1-x^2)(1-\frac{x^2}{4})(1-\frac{x^2}{9})\ldots\ldots\ldots\ldots\ldots(15)$

Hence, $2\cos\frac{1}{2}\pi x.\,\sin\frac{1}{2}\pi x=\sin\pi x=\pi x(1-\frac{x^2}{4})(1-\frac{x^2}{16})(1-\frac{x^2}{36})\ldots\ldots$

$$\times(1-x)(1-\frac{x^2}{9})(1-\frac{x^2}{25})\ldots\ldots$$

$$(15)=2\sin\tfrac{1}{2}\pi x.\,(1-x^2)(1-\frac{x^2}{9})(1-\frac{x^2}{25})\ldots\ldots$$

$$\therefore \cos\tfrac{1}{2}\pi x=(1-x^2)(1-\frac{x^2}{9})(1-\frac{x^2}{25})\ldots\ldots\ldots\ldots(16)$$

From (13) and (15) we have

$$\Gamma(x+1)=\frac{\pi x}{\sin\pi x}\times\text{limit of } n^x\,(1-x)(1-\tfrac{1}{2}x)(1-\tfrac{1}{3}x)\ldots\ldots(1-\frac{1}{n}x)$$

$$=\frac{\pi x}{\sin\pi x}.\,\frac{1-x}{1^x.2^{-x}}.\,\frac{1-\frac{1}{2}x}{2^x.3^{-x}}.\,\frac{1-\frac{1}{3}x}{3^x.4^{-x}}\ldots\ldots\ldots\ldots(17)$$

By (13), $\Gamma(x+1)=$ limit of $\dfrac{(2n)^x}{(1+x)(1+\frac{1}{2}x)\ldots\ldots(1+\frac{1}{2n}x)}$, writing $2n$ for n, which is allowable;

also $\Gamma(\frac{x}{2}+1)=$ limit of $\dfrac{n^{\frac{x}{2}}}{(1+\frac{x}{2})(1+\frac{x}{4})\ldots\ldots(1+\frac{x}{2n})}$

$$\therefore\ \frac{\Gamma(x+1)}{\Gamma(\frac{x}{2}+1)}=2^x.\ \text{limit of}\ \frac{n^{\frac{x}{2}}}{(1+x)(1+\frac{1}{3}x)\ldots\ldots(1+\frac{1}{2n-1}x)}\ldots\ldots(18)$$

Hence also,

$$\frac{\Gamma(x+1)}{\{\Gamma(\frac{x}{2}+1)\}^2}=2^x.\ \text{limit of}\ \frac{1+\frac{x}{2}}{1+x}.\,\frac{1+\frac{x}{4}}{1+\frac{x}{3}}.\,\frac{1+\frac{x}{6}}{1+\frac{x}{5}}\ldots\ldots\frac{1+\frac{x}{2n}}{1+\frac{x}{2n-1}}$$

$$\therefore\frac{1+\frac{x}{2}}{1+x}.\,\frac{1+\frac{x}{4}}{1+\frac{x}{3}}.\,\frac{1+\frac{x}{6}}{1+\frac{x}{5}}\ldots\ldots=2^{-x}\frac{\Gamma(x+1)}{\{\Gamma(\frac{x}{2}+1)\}^2}\ldots\ldots\ldots\ldots(19)$$

Multiply (16) and (19),

$$\therefore(1-x)(1+\frac{x}{2})(1-\frac{x}{3})(1+\frac{x}{4})\ldots=2^{-x}.\,\cos\tfrac{1}{2}\pi x.\,\frac{\Gamma(x+1)}{\{\Gamma(\frac{x}{2}+1)\}^2}\ldots(20)$$

In (20) write $-x$ for x, and divide (20) by the result,

$$\therefore \frac{1-x}{1+x}\cdot\frac{1+\frac{x}{2}}{1-\frac{x}{2}}\cdot\frac{1-\frac{x}{3}}{1+\frac{x}{3}}\cdot\frac{1+\frac{x}{4}}{1-\frac{x}{4}}\ldots\ldots=4^{-x}\left\{\frac{\Gamma(-\frac{x}{2}+1)}{\Gamma(\frac{x}{2}+1)}\right\}^2\cdot\frac{\Gamma(x+1)}{\Gamma(-x+1)}$$

$$=4^{-x}\cdot\frac{\frac{1}{2}\pi x}{\tan\frac{1}{2}\pi x}\cdot\frac{\{\Gamma(x+1)\}^2}{\{\Gamma(\frac{x}{2}+1)\}^4},\text{ by (3)}\ldots\ldots\ldots\ldots(21)$$

Scholium.—The equations 1—7, 15, and 16 are not new; but the mode of investigation is, so far as I know, original. With respect to the others, I am not aware that they have been noticed; and even should they have been anticipated, they can be but very little known; and hence their publication may be compatible with the objects of the Mathematician.

SOLUTIONS OF MATHEMATICAL EXERCISES.

VIII.—*By ϕ.*

If the circumference of a circle be divided into six equal parts in the points A_1, $A_2 \ldots . A_6$, and if A_1A_3 be joined intersecting the radius OA_2 in B_1; B_1A_4 joined cutting OA_3 in B_2; B_2A_5 joined cutting OA_4 in B_3; and so on: then OB_1, OB_2, *etc.*, are respectively the half, third, fourth, *etc.* parts of the radius.

[First Solution.—*Mr. Samuel Bills, Hawton.*]

(The student will readily supply the figure.)

Draw the chords A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1; then since $OA_1A_2A_3$ is a rhombus, it is evident that OB_1 will be equal to half the radius. Also, since $OA_2A_3A_4$ is a rhombus, OB_1 and A_3A_4 will be parallel lines; therefore the triangles OB_1B_2 and $B_2A_3A_4$ will be similar, and since the line OB_2 evidently bisects the angle B_1OA_4, we shall have

$$OB_2 : B_2A_3 : : B_1B_2 : B_2A_4 : : OB_1 : OA_4.$$

But $OB_1 = \frac{1}{2}OA_4$; therefore $OB_2 = \frac{1}{2}B_2A_3$, and consequently $OB_1 = \frac{1}{3}OA_3$ = one-third part of the radius. In exactly the same manner it may be proved that $OB_3 = \frac{1}{4}$ radius; $OB_4 = \frac{1}{5}$ radius; $OB_5 = \frac{1}{6}$ radius; and so on.

[Second Solution.—*Mr. John Laws, Newcastle-on-Tyne.*]

Produce A_4B_1, A_5B_2, *etc.*, to meet the chords A_1A_2, A_2A_3, *etc.*, in the points B, C, D, E, F respectively. Now $OB_1 = \frac{1}{2}A_1A_6 = \frac{1}{2}OA_1$. Also $A_1B = 2OB_2$; $BA_2 = OB_2$; $OA_1 = A_1A_2 = A_1B + BA_2 = 3OB_2$; therefore $OB_2 = \frac{1}{3}OA_1$. Again $A_2C = 2OB_3$, and because $A_3B_2 = 2OB_2$, therefore $CA_3 = 2OB_3$; hence $OA_1 = A_2C + CA_3 = 4OB_3$ $\therefore OB_3 = \frac{1}{4}OA_1$. Similarly $OB_4 = \frac{1}{5}OA_1$; $OB_5 = \frac{1}{6}OA$; and $OB_6 = \frac{1}{7}OA_1$.

[Third Solution.—*Mr. J. W. Elliott, Greatham; and similarly by Mr. A. Hills, Milfield, near Wooler.*]

Let $R =$ the radius of the circle; then the figure $OA_1A_2A_3$ is a rhombus, and the diagonals OA_2, A_1A_3 bisect each other; wherefore $OB_1 = \frac{1}{2}OA_2 = \frac{R}{2}$.

Again, because the angle $B_1OB_2 = B_2A_3A_4$, and the angles at B_2 equal, it follows that the triangles B_1OB_2, $B_2A_3A_4$ are similar, therefore

$$OB_1 : OB_2 :: A_3A_4 :: A_3B_2 ; \text{ but}$$

$$OB_1 = \tfrac{1}{2}A_3A_4 \therefore OB_2 = \tfrac{1}{3}A_3B_2 = \frac{R}{3}.$$

And similarly may it be shewn, that

$$OB_3 = \tfrac{1}{3}B_3A_4 = \frac{R}{4};\quad OB_4 = \tfrac{1}{4}B_4A_5 = \frac{R}{5};$$

and hence for the n^{th} intersection $OB_n = \dfrac{R}{n+1}$.

Cor. 1.—$A_1B_1 = \frac{1}{2}A_1A_3$, $B_1B_2 = \frac{1}{3}B_1A_4$, $B_2B_3 = \frac{1}{4}B_2A_5$, *etc.*, which follow from the proposition.

Cor. 2.—Join A_1A_5, A_3A_5: then A_2A_6 is trisected by A_1A_3, A_1A_5 in the points O_1, O_2. For since the arc $A_2A_3 = A_1A_6$, the angle $A_2A_1A_3 = A_1A_2A_6$, and therefore $A_1O_1 = O_1A_2$. In like manner $A_1O_2 = O_2A_6$. Moreover, since the triangle $A_1A_3A_5$ is equilateral, and A_2A_6 parallel to A_3A_5, it follows that $A_1O_1 = O_1O_2 = O_2A_1$; that is, $A_2O_1 = O_1O_2 = O_2A_6$.

[Fourth Solution.—*Mr. James Anderson, Montrose.*]

Let r = the radius, and assume that the proposition is true in the case of OB_n; that is, let $OB_n = \dfrac{r}{n+1}$; then in the triangle $B_n\, OA_{n+3}$, the angle at O measures $\dfrac{2\pi}{3}$, and is bisected by the line OB_{n+1}. To determine OB_{n+1}, let $OB_{n+1}B_n = \theta$; then from the triangles $OA_{n+3}\,B_{n+1}$ and $OB_n\,B_{n+1}$ we have

$$OB_{n+1} = r.\ \frac{\sin(\theta - \frac{1}{3}\pi)}{\sin\theta} = r\,(\cos\frac{\pi}{3} - \cot\theta\sin\frac{\pi}{3});$$

$$OB_{n+1} = \frac{r}{n+1}\cdot\frac{\sin(\theta+\frac{1}{3}\pi)}{\sin\theta} = \frac{r}{n+1}\,(\cos\frac{\pi}{3} + \cot\theta\sin\frac{\pi}{3});$$

hence $\qquad (n+1)\,(\cos\dfrac{\pi}{3} - \cot\theta\sin\dfrac{\pi}{3}) = \cos\dfrac{\pi}{3} + \cot\theta\sin\dfrac{\pi}{3},$

and $\qquad \cot\theta = \dfrac{n}{n+2}\,\dfrac{\cos\frac{1}{3}\pi}{\sin\frac{1}{3}\pi}$ .... (1)

Consequently $OB_{n+1} = r\,(\cos\dfrac{\pi}{3} - \cot\theta\sin\dfrac{\pi}{3}) = r\,(\dfrac{1}{2} - \dfrac{1}{2}\cdot\dfrac{n}{n+2}) = \dfrac{r}{n+2}$;

hence if the proposition be true for one value of n, it is true for the next value, and consequently for every succeeding value. But it is easy to see from the triangle A_1OA_3 that it is true for $n = 1$, and therefore it is true universally.

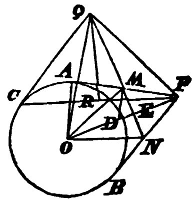
Cor.—From (1) it appears that when the vertical angle of a triangle is 120°, the line which bisects it cuts the base of the triangle at an angle more and more approaching to 60°, as one of the sides becomes larger and larger than the other.

Correct and elegant solutions were also received from Messrs. Thomas Dobson, Totteridge; W. H. Levy, Shaftbourne; Robert Rawson, Manchester; T. Weddle, Newcastle; and Theta.

IX.—*Mr. Fenwick.*

From a point P without a circle draw the tangents PA, PB, and also any line PC to the concave circumference: then the tangent at C, a perpendicular bisecting PC, and the line which joins the middle points of PA, PB, tend to the same point.

[First Solution.—*Mr. Weddle, Newcastle on Tyne.*]



Let PA, PB and PC be bisected in M, N and R, and let QR perpendicular to PC meet the tangent at C in Q. Then the other lines being drawn as in the figure,

$$OQ^2 - QP^2 = OQ^2 - QC^2 = OC^2:$$

and $$OM^2 - PM^2 = OM^2 - AM^2 = OC^2.$$

Hence $$OQ^2 - QP^2 = OM^2 - PM^2 \text{ or } ON^2 - PN^2.$$

It follows, therefore, that M, Q, N are in a straight line perpendicular to OP. Q. E. D.

If PC meet the convex circumference in C′, and PC′ be bisected perpendicularly by Q′R′ meeting the tangent at C′ in Q′, it may in like manner be shown that M, N, Q′ are in a straight line. Hence we easily deduce the following property:

From any point P without a circle draw any straight line PC′C, cutting the circle in C, C′. Let the tangent at C and perpendicular bisecting PC meet in Q, and the tangent at C′ and perpendicular bisecting PC′ meet in Q′; then the straight line joining QQ′ bisects the tangents PA, PB drawn from P.

[Second Solution.—*Analyticus, and similarly by Mr. John Laws.*]

Take O the centre of the circle for the origin of rectangular co-ordinates, and the line joining OP for the axis of x. Then the line joining the points of contact A, B, and that through the points of bisection of the tangents at A and B are each perpendicular to the axis of x, and consequently parallel to each other. Let r be the radius of the circle, and put $OP = a$; then denoting the points A and C respectively by x_1y_1 and x_2y_2, the equation of the tangent AP is $yy_1 + xx_1 = r^2$, which, when $y = 0$, gives

$$OP = \frac{r^2}{x_1} = a \quad \therefore \quad ax_1 = r^2 \ldots\ldots\ldots\ldots\ldots (1)$$

Hence the equation of MN is $x = \dfrac{a + x_1}{2}$ ....(2)

Again the equations of PC and the tangent at C are respectively

$$y(a - x_2) - y_2(a - x) = 0 \ldots (3) \qquad yy_2 + xx_2 = r^2 \ldots\ldots\ldots (4)$$

But the co-ordinates of the middle point of PC are

$$x = \frac{a + x_2}{2} \text{ and } y = \frac{y_2}{2},$$

and therefore the equation of the line perpendicular to PC through its middle point is

$$y = \frac{a - x_2}{y_2}x - \frac{a^2 - r^2}{2y_2} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Writing in equations (4) and (5) the value of x in (2), and keeping in mind the relation in (1), we have from either the same value of y, viz.

$$y=\frac{a(x_1-x_2)+x_1(a-x_2)}{2y_2}$$

which proves the truth of the theorem, and at the same time furnishes the co-ordinates of the common point of intersection of the three specified lines.

Again, from the equation of the circle, and that of the line PC, we have, for the co-ordinates of the point where the line PC cuts the convex circumference, the values

$$x=\frac{2ar^2-x_2(a^2+r^2)}{a^2+r^2-2ax_2} \text{ and } y=\frac{y_2(a^2-r^2)}{a^2+r^2-2ax_2};$$

and consequently the equation of the tangent at that point is

$$yy_2(a^2-r^2)+x\{2ar^2-x_2(a^2+r^2)\}=r^2(a^2+r^2-2ax_2)\ldots\ldots(6)$$

Writing in (4) and (6) x_1 for x, and remembering the relation $ax_1=r^2$, we find the same value of y from either of these equations, viz.

$$y=\frac{r^2-x_1x_2}{y_2}$$

and hence the tangents at the points of intersection of the line which cuts the circle, and the line joining the points of contact A, B, meet in the same point whose co-ordinates are

$$x=x_1 \text{ and } y=\frac{r^2-x_1x_2}{y_2}.$$

Hence also this property:

From a point P without a circle draw the tangents PA, PB, and also any line PCC′ to meet the circumference in C and C′; then the tangents at C and C′, and the chord of contact AB, tend to the same point.

[Third Solution—*Mr. A. Hills, Milfield.*]

Let the tangent at the point C, and MN the line bisecting PA, PB, meet in Q. Join the point P and the centre of the circle O; and from M draw MD at right angles to PA, meeting PO in D. Then it will be obvious that PO, MN intersect perpendicularly in E, that PO is bisected in D, and that MD is half the radius of the given circle.

Hence, joining AO, QO, CO, and QP, we have (*Euc.* i. 47,)

$$\begin{aligned}QP^2 &= QE^2+EP^2=QO^2-OE^2+EP^2\\ &= QC^2+CO^2-(OE^2-EP^2)\\ &= QC^2+AO^2-2OP.DE\\ &= QC^2+AO^2-4DP.DE\\ &= QC^2+AO^2-4DM^2\ (\textit{Euc.}\ \text{vi. } 8)\\ &= QC^2,\ \text{since } AO=2DM.\end{aligned}$$

Consequently, QP = QC, and therefore a perpendicular bisecting PC will tend to the point Q. Q. E. D.

Good solutions were also sent by Messrs. Thomas Dobson, Totteridge, Herts; John Laws, Newcastle-on-Tyne; Robert Rawson, Manchester; and Theta.

X.—*By E. I. F., London.*

Given the system of equations,

$$\begin{aligned} a &= a \\ a + x &= a a_1 \\ a + 2x &= a a_1^2 a_2 \\ a + 3x &= a a_1^3 a_2^2 a_3 \\ &\cdots\cdots\cdots \end{aligned}$$

to find the law of successive developement, and the mutual relation of the two series.

[SOLUTION.—*Pen-and-Ink.*]

Divide the second equation by the first, the third by the second, and so on: then divide these quotient-equations, the second by the first, the third by the second, and so on. There thus result successively the values of a_1, a_2, $\ldots\ldots a_n$.

$$\frac{a+x}{a} = a_1 \qquad\qquad \frac{a(a+2x)}{(a+x)^2} = a_2$$

$$\frac{a+2x}{a+x} = a_1 a_2 \qquad\qquad \frac{(a+x)(a+3x)}{(a+2x)^2} = a_3$$

$$\frac{a+3x}{a+2x} = a_1 a_2 a_3 \qquad\qquad \frac{(a+2x)(a+4x)}{(a+3x)^2} = a_4$$

$$\frac{a+4x}{a+3x} = a_1 a_2 a_3 a_4 \qquad\qquad \frac{(a+3x)(a+5x)}{(a+4x)^2} = a_5$$

$$\cdots\cdots\cdots \qquad\qquad \cdots\cdots\cdots$$

the $(n+1)^{\text{th}}$ term of which is, obviously,

$$\frac{\{a+(n-1)x\}\,\{a+(n+1)x\}}{(a+nx)^2}.$$

⁂ We are aware that the learned proposer of this question contemplated an important application of this series of equations, and we hope he will furnish us with a short paper on the subject.—EDS.

XII.—*W. F., Durham.*

The six co-ordinate axes of any two systems of rectangular co-ordinates in space (the origin being the same) lie in a conical surface of the second degree: also the six co-ordinate planes touch a conical surface of the second degree.

[SOLUTION.—*Mr. Weddle.*]

Take one of the systems of rectangular axes for those of reference, and let the equations of the other system be

$$\left.\begin{aligned} y &= \alpha z \\ x &= \beta z \end{aligned}\right\}\ldots(1) \qquad \left.\begin{aligned} y &= \alpha' z \\ x &= \beta' z \end{aligned}\right\}\ldots(2) \qquad \left.\begin{aligned} y &= \alpha'' z \\ x &= \beta'' z \end{aligned}\right\}\ldots(3)$$

Now it is obvious that

$$(y-\alpha z)(x-\beta' z) + \mathrm{A}\,(y-\alpha' z)(x-\beta z) = 0 \ldots\ldots\ldots (4)$$

represents a cone of the second order passing through the lines (1, 2); it also passes through the axes of x and y, for equation (4) is satisfied by any value of x, when $y = 0$, and $z = 0$, and by any value of y, when $x = 0$, and $z = 0$; in order that (4) may pass through the axis of z, (4) must be satisfied by any value of z, when $x = 0$, and $y = 0$, this requires that

$A = -\frac{\alpha\beta'}{\alpha'\beta}$. This value being substituted in (4), gives after reduction and multiplication by β''

$$(\alpha'\beta - \alpha\beta')\beta'' xy - \alpha\alpha'(\beta - \beta')\beta'' xz + \beta\beta'\beta''(\alpha - \alpha')yz = 0 \ldots\ldots (5)$$

which is the equation of the cone passing through the axes and the lines (1, 2).

Moreover, since the lines (1, 2, 3) are perpendicular to each other,—(*Young's Anal. Geom.* vol. ii. p. 130)

$$\alpha\alpha' + \beta\beta' + 1 = 0 \ldots\ldots (6)$$

$$\alpha\alpha'' + \beta\beta'' + 1 = 0 \ldots\ldots (7)$$

$$\alpha'\alpha'' + \beta'\beta'' + 1 = 0 \ldots\ldots (8)$$

Deduct (8) from (7), $\therefore\ (\alpha - \alpha')\alpha'' = -(\beta - \beta')\beta''$.

Multiply (7) by α', and (8) by α, and deduct,

$$\therefore\ (\alpha'\beta - \alpha\beta')\beta'' = \alpha - \alpha'.$$

Substitute these values in (5), and divide by $\alpha - \alpha'$

$$\therefore\ xy + \alpha\alpha'\alpha'' xz + \beta\beta'\beta'' yz = 0 \ldots\ldots (9)$$

This then is the equation of the cone of the second degree passing through the axes and the lines (1, 2); but (9) being symmetrical with respect to $\alpha\alpha'\alpha''$ as well as $\beta\beta'\beta''$, would, it is evident, equally result from employing any two of the equations (1, 2, 3); that is, the cone represented by (9) passes through the axes and the three lines (1, 2, 3), which establishes the first part of the exercise.

Again, $z^2 + A^2y^2 + B^2x^2 + 2Cxy + 2Dzx + 2Eyz = 0 \ldots\ldots (10)$ is the equation of a cone whose vertex is the origin. The traces on the planes of zy, zx, and yx are respectively

$$z^2 + A^2y^2 + 2Ezy = 0 \ldots\ldots (11)$$

$$z^2 + B^2x^2 + 2Dzx = 0 \ldots\ldots (12)$$

$$A^2y^2 + B^2x^2 + 2Cyx = 0 \ldots\ldots (13)$$

Now each of these equations represents two straight lines, but if the conical surface (10) touches the co-ordinate planes, these two straight lines must coincide, which requires that the left hand members of (11), (12), and (13) be perfect squares;

$$\therefore\ A^2 = E^2,\quad B^2 = D^2,\quad \text{and}\quad A^2B^2 = C^2.$$

Hence $E = A$, $D = B$, and $C = \pm AB$, if the upper sign in this value of C be taken, (10) will be found to represent a plane, take therefore the lower sign and substitute in (10);

$$\therefore\ z^2 + A^2y^2 + B^2x^2 + 2Azy + 2Bzx - 2AByx = 0 \ldots\ldots (14)$$

is the equation of a cone touching the co-ordinate planes.

Let the equations of the second system of co-ordinate planes be

$$z = ay + bx \ldots\ldots (15)$$

$$z = a'y + b'x \ldots\ldots (16)$$

$$z = a''y + b''x \ldots\ldots (17)$$

Since these are perpendicular we have (*Young's Anal. Geom.*, vol ii., p. 145),

$$aa' + bb' + 1 = 0 \ldots\ldots (18)$$

$$aa'' + bb'' + 1 = 0 \ldots\ldots (19)$$

$$a'a'' + b'b'' + 1 = 0 \ldots\ldots (20)$$

Eliminate z from (14) by means of (15), and we have

$$\therefore \ (\mathrm{A}+a)^2y^2+(\mathrm{B}+b)^2x^2+2(ab+\mathrm{A}b+\mathrm{B}a-\mathrm{AB})yx=0\ldots(21)$$

for the intersection of the plane (15) and surface (14); if they be tangent, (21) must be a square, which requires that either

$$(\mathrm{A}+a)(\mathrm{B}+b)=ab+\mathrm{A}b+\mathrm{B}a-\mathrm{AB}, \text{ or}$$

$$-(\mathrm{A}+a)(\mathrm{B}+b)=ab+\mathrm{A}b+\mathrm{B}a-\mathrm{AB},$$

the former of these gives $\mathrm{A}=0$ or $\mathrm{B}=0$, and is therefore to be rejected, but the latter reduces to

$$ab+\mathrm{A}b+\mathrm{B}a=0 \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

Similarly, $$a'b'+\mathrm{A}b'+\mathrm{B}a'=0\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

and, $$a''b''+\mathrm{A}b''+\mathrm{B}a''=0\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

Eliminate B from (22) and (23), and multiply by a''

$$\therefore \ aa'a''\,(b-b')+\mathrm{A}a''(a'b-ab')=0\ldots\ldots\ldots\ldots(25)$$

Deduct (20) from (19), also a times (20) from a' times (19)

$$\therefore \ (a-a')a''=-b''(b-b') \text{ and } (a'b-ab')b''=a-a'$$

$$\therefore \ (a'b-ab')a''=\frac{a''}{b''}(a-a')=-(b-b').$$

Substitute this in (25), and divide by $b-b'$,

$$\therefore \ \left.\begin{array}{l}\mathrm{A}=aa'a'' \\ \mathrm{B}=bb'b''\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

Similarly,

These values being symmetrical, we conclude as before that they satisfy (22), (23) and (24). Hence, by (14) and (26), the cone whose equation is

$$z^2+a^2a'^2a''^2y^2+b^2b'^2b''^2x^2+2aa'a''zy+2bb'b''zx-2aa'a''bb'b''yx=0,$$

touches both systems of rectangular co-ordinate planes.

Solutions were also received from the proposer and Theta.

XIII—*Mr. R. H. Wright, London.*

If S be the focus of an ellipse and the centre of force, Pp a chord through S; then the time of a body revolving from P to p is

$$t=\frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}(2m-\sin 2m),$$

where $\tan m=\dfrac{\tan e}{\sin\theta}$, angle ASP $=\theta$, and eccentricity $=\cos e$.

[First Solution.—*Mr. Weddle, Newcastle.*]

By Kepler's first law, or *Young's Mechanics*, p. 209, we have

$$r^2d\theta=cdt\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where c is a constant quantity. Now, writing $\cos e$ for e, the polar equation of the ellipse is

$$r=\frac{a\sin^2 e}{1+\cos e\cos\theta};$$

Hence, by substitution and integration, we get

$$t=\frac{a^2\sin^4 e}{c}\int\frac{d\theta}{(1+\cos e\cos\theta)^2}$$

$$t=\frac{a^2\sin^2 e}{c}\left\{\frac{2}{\sin e}\tan^{-1}\left(\tan\frac{e}{2}\tan\frac{\theta}{2}\right)-\frac{\cos e\sin\theta}{1+\cos e\cos\theta}\right\}.$$

This, taken between the limits $-(\pi-\theta)$ and θ, gives after reduction,

$$t=\frac{a^2\sin^2 e}{c}\left\{\frac{2}{\sin e}\tan^{-1}\frac{\tan e}{\sin\theta}-\frac{2\cos e\sin\theta}{1-\cos^2 e\cos^2\theta}\right\}.$$

Let $\tan m=\frac{\tan e}{\sin\theta}$; therefore $t=\frac{a^2\sin e}{c}(2m-\sin 2m)$............(2)

Again, if R be the attractive force, we have (*Young's Mechanics* 211,)

$$R=\frac{c^2}{r^2}\left\{\frac{d^2(r^{-1})}{d\theta^2}+\frac{1}{r}\right\}=\frac{c^2}{a\sin^2 e}\cdot\frac{1}{r^2}.$$

Now if μ be the value of R when $r=1$, we have $c=a^{\frac{1}{2}}\mu^{\frac{1}{2}}\sin e$; hence by (2)

$$t=\frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}(2m-\sin 2m).$$

[SECOND SOLUTION.—*Mr. James Anderson.*]

Instead of solving this exercise by means of the equation which connects the time and the true anomaly in an elliptical orbit, it may not be amiss to deduce it as a corollary from a more general proposition by Laplace. In chap. iv. liv. ii., he has shown that

$$t=\frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}\{\cos^{-1}z'-\cos^{-1}z-\sin(\cos^{-1}z')+\sin(\cos^{-1}z)\}\text{ ....(1)}$$

where $z=\frac{2a-(r+r')+c}{2a}$, and $z'=\frac{2a-(r+r')-c}{2a}$;

a being the semi-major axis, c a chord of the ellipse, r and r' the radii vectores of its extremities, and t the time of describing the arc cut off by the chord.

In the present exercise $c=r+r'$; and hence

$$z=1;\ z'=\frac{a-(r+r')}{a};\ \cos^{-1}z=0,\ \sin(\cos^{-1}z)=0.$$

By the polar equation of the ellipse $r=\frac{a\sin^2 e}{1+\cos e\cos\theta}$, and since by writing $\pi-\theta$ for θ, we obtain r', we have

$$r'=\frac{a\sin^2 e}{1-\cos e\cos\theta},\text{ and hence } z'=\frac{\cos^2 e\sin^2\theta-\sin^2 e}{1-\cos^2 e\cos^2\theta}.$$

Assume $z'=\cos 2m$, then we have

$$\frac{\cos^2 e\sin^2\theta-\sin^2 e}{1-\cos^2 e\cos^2\theta}=\cos 2m=2\cos^2 m-1;$$

hence $\cos^2 m=\frac{\cos^2 e\sin^2\theta}{1-\cos^2 e\sin^2\theta}$, and $\sin^2 m=\frac{\sin^2 e}{1-\cos^2 e\cos^2\theta}$;

therefore $\tan m=\frac{\tan e}{\sin\theta}$, and $\cos^{-1}z'=2m$, and $\sin(\cos^{-1}z')=\sin 2m$; hence by Laplace's theorem

$$t=\frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}(2m-\sin 2m).$$

[THIRD SOLUTION.—*Mr. R. H. Wright, the Proposer.*]

By the first law of Kepler we have the relation $r^2 d\theta = c dt$, where $c = a^{\frac{1}{2}}\mu^{\frac{1}{2}}\sin e$, in the notation of the question. Now writing $\cos e$ for e in the polar equation of the ellipse, we have

$$r = \frac{a \sin^2 e}{1 + \cos e \cos \theta}.$$

Substituting these values of c and r in Kepler's equation, and putting $n^2\mu = a^3$, we have

$$ndt = \frac{\sin^3 e \,.\, d\theta}{(1 + \cos e \cos \theta)^2};$$

$$\therefore \quad nt = 2\tan^{-1}\left(\tan\frac{e}{2}\ \tan\frac{\theta}{2}\right) - \frac{\sin e \cos e \sin\theta}{1 + \cos e \cos\theta} \dots\dots..(1)$$

Writing $\pi - \theta$ for θ in (1), we get

$$nt = 2\tan^{-1}\left(\tan\frac{e}{2}\ \cot\frac{\theta}{2}\right) - \frac{\sin e \cos e \sin\theta}{1 - \cos e \cos\theta} \dots\dots..(2)$$

To obtain the whole time we must take the sum of these, and recollecting the formula

$$\tan^{-1} t_1 + \tan^{-1} t_2 = \tan^{-1}\frac{t_1 + t_2}{1 - t_1 t_2},$$

we have readily

$$nt = 2\tan^{-1}\left\{\frac{\tan\frac{1}{2}e}{1 - \tan^2\frac{1}{2}e}\cdot\left(\tan\tfrac{1}{2}\theta + \cot\tfrac{1}{2}\theta\right)\right\} - \frac{\sin 2e \sin\theta}{1 - \cos^2 e \cos^2\theta}$$

$$= 2\tan^{-1}\frac{\tan e}{\sin\theta} - \frac{\sin 2e \sin\theta}{1 - \cos^2 e \cos^2\theta}$$

$$= 2m - \sin 2m, \text{ when } \tan m = \frac{\tan e}{\sin\theta}.$$

XIV.—*Mr. Rutherford.*

Let r, r_1, r_2, r_3 be the radii of four circles in mutual contact either on a sphere, or in a plane, of which the circle radius r is touched externally by the other three; then if d and d_1 are the respective distances of the centres of the circles radii r and r_1 from the great circle or line joining the centres of the circles radii r_2 and r_3, we shall have

ON THE SPHERE.	ON THE PLANE.
$\dfrac{\sin d}{\sin r} - \dfrac{\sin d_1}{\sin r_1} = 2$	$\dfrac{d}{r} - \dfrac{d_1}{r_1} = 2.$

[FIRST SOLUTION.—*Mr. Weddle, Newcastle.*]

Let O, O_1, O_2, O_3 be the centres of the circles radii r, r_1, r_2, r_3 respectively. Then as will be shown in the solution to the next exercise, $OO_2 + OO_3 + O_2O_3$ being $= 2S$,

$$\sin d = \frac{2\sqrt{\sin S \sin(S - OO_2)\sin(S - OO_3)\sin(S - O_2O_3)}}{\sin O_2O_3}.$$

But $OO_2 = r + r_2$, $OO_3 = r + r_3$, and $O_2O_3 = r_2 + r_3$; hence

$$\frac{\sin d}{\sin r} = \frac{2\sqrt{\sin(r + r_2 + r_3)\sin r \sin r_2 \sin r_3}}{\sin(r_2 + r_3)\sin r}$$

$$= \frac{2\sqrt{\cot r \cot r_2 + \cot r \cot r_3 + \cot r_2 \cot r_3 - 1}}{\cot r_2 + \cot r_3}.$$

Now, Mathematician, No. II., p. 102, eq. 8,

$$2\sqrt{\cot r \cot r_2 + \cot r \cot r_3 + \cot r_2 \cot r_3 - 1} = \cot r - \cot r_1 + \cot r_2 + \cot r_3;$$

$$\therefore \quad \frac{\sin d}{\sin r} = \frac{\cot r - \cot r_1 + \cot r_2 + \cot r_3}{\cot r_2 + \cot r_3} \dots\dots\dots\dots\dots\dots (1)$$

In a similar manner, recollecting that

$$2\sqrt{\cot r_1 \cot r_2 + \cot r_1 \cot r_3 + \cot r_2 \cot r_3 - 1} = \cot r - \cot r_1 - \cot r_2 - \cot r_3,$$

we shall have

$$\frac{\sin d_1}{\sin r_1} = \frac{\cot r - \cot r_1 - \cot r_2 - \cot r_3}{\cot r_2 + \cot r_3} \dots\dots\dots\dots\dots\dots (2)$$

Deduct (2) from (1), and we obtain at once

$$\frac{\sin d}{\sin r} - \frac{\sin d_1}{\sin r_1} = 2.$$

The investigation on the plane is similar, writing d, r, *etc.*, instead of $\sin d$, $\sin r$, *etc.*, and $\frac{1}{r}$, $\frac{1}{r_1}$, *etc.*, instead of $\cot r$, $\cot r_1$, *etc.*, and taking away -1 from under the radical.

[SECOND SOLUTION.—*Mr. James Anderson.*]

By the known property of the perpendicular of a spherical triangle, we have

$$\sin^2 d \sin^2(r_2 + r_3) = 4 \sin r \sin r_2 \sin r_3 \sin(r + r_2 + r_3);$$

and hence expanding both sides of the equation, and reducing the result, we get

$$\frac{\sin^2 d}{\sin^2 r} = \frac{4(\cot r \cot r_2 + \cot r \cot r_3 + \cot r_2 \cot r_3 - 1)}{(\cot r_2 + \cot r_3)^2}.$$

Similarly, $$\frac{\sin^2 d_1}{\sin^2 r_1} = \frac{4(\cot r_1 \cot r_2 + \cot r_1 \cot r_3 + \cot r_2 \cot r_3 - 1)}{(\cot r_2 + \cot r_3)^2}.$$

Therefore $$\frac{\sin^2 d}{\sin^2 r} + \frac{\sin^2 d_1}{\sin^2 r_1}$$

$$= \frac{4(\cot r \cot r_2 + \cot r \cot r_3 + \cot r_1 \cot r_2 + \cot r_1 \cot r_3 + 2\cot r_2 \cot r_3 - 2)}{(\cot r_2 + \cot r_3)^2} \dots (1)$$

and $$\frac{\sin^2 d}{\sin^2 r} - \frac{\sin^2 d_1}{\sin^2 r_1} = \frac{4(\cot r - \cot r_1)(\cot r_2 + \cot r_3)}{(\cot r_2 + \cot r_3)^2} = \frac{4(\cot r - \cot r_1)}{\cot r_2 + \cot r_3} \dots (2)$$

Then by the second equation preceding that marked (1) in page 99 of the Mathematician,

$$2(\cot r\cot r_2+\cot r\cot r_3+\cot r_1\cot r_2+\cot r_1\cot r_3+2\cot r_2\cot r_3-2)$$
$$=\cot^2 r+\cot^2 r_1+\cot^2 r_2+\cot^2 r_3+2\cot r_2\cot r_3-2\cot r\cot r_1$$
$$=(\cot r-\cot r_1)^2+(\cot r_2+\cot r_3)^2;\text{ and therefore}$$
$$\frac{\sin^2 d}{\sin^2 r}+\frac{\sin^2 d_1}{\sin^2 r_1}=2\left\{1+\frac{(\cot r-\cot r_1)^2}{(\cot r_2+\cot r_3)^2}\right\}=2+\tfrac{1}{8}\left(\frac{\sin^2 d}{\sin^2 r}-\frac{\sin^2 d_1}{\sin^2 r_1}\right)^2 \ldots (3)$$

We have therefore connected those quantities among which it is desired to to establish a relation, and it only remains to reduce this relation to a simpler form. In order to this, add $\dfrac{\sin^2 d}{\sin^2 r}-\dfrac{\sin^2 d_1}{\sin^2 r_1}$ to both sides of (3) and divide by 2; there results

$$\frac{\sin^2 d}{\sin^2 r}=1+\tfrac{1}{2}\left(\frac{\sin^2 d}{\sin^2 r}-\frac{\sin^2 d_1}{\sin^2 r_1}\right)+\frac{1}{16}\left(\frac{\sin^2 d}{\sin^2 r}-\frac{\sin^2 d_1}{\sin^2 r_1}\right)^2,$$

and extracting the square root of both sides,

$$\pm\frac{\sin d}{\sin r}=1+\tfrac{1}{4}\left(\frac{\sin^2 d}{\sin^2 r}-\frac{\sin^2 d_1}{\sin^2 r_1}\right)\ldots\ldots\ldots\ldots (4)$$

But $\dfrac{\sin d}{\sin r}$ is essentially positive, and when r is less than r_1, as will be the case, when the circle radius r is touched externally, it is easy to see from (2) that $\dfrac{\sin^2 d}{\sin^2 r}-\dfrac{\sin^2 d_1}{\sin^2 r_1}$ is positive; hence it is necessary to take the upper sign in the preceding equation. Arranging the terms,

$$\frac{\sin^2 d}{\sin^2 r}-4\,\frac{\sin d}{\sin r}+4=\frac{\sin^2 d_1}{\sin^2 r_1};$$

hence $\quad \dfrac{\sin d}{\sin r}-2=\pm\dfrac{\sin d_1}{\sin r_1}$, or $\dfrac{\sin d}{\sin r}\mp\dfrac{\sin d_1}{\sin r_1}=2\ldots\ldots\ldots\ldots (5)$

It may be found from the value of $\dfrac{\sin^2 d}{\sin^2 r}$, by the ordinary method of maxima and minima, that its least possible value is 1, and that there is no maximum. The same is true of $\dfrac{\sin^2 d_1}{\sin^2 r_1}$, and hence we are limited to the upper sign in the last equation, or

$$\frac{\sin d}{\sin r}-\frac{\sin d_1}{\sin r_1}=2.$$

Perhaps the proper sign in (4, 5) is sufficiently evident from the consideration, that $\dfrac{\sin^2 d}{\sin^2 r}$ and $\dfrac{\sin^2 d_1}{\sin^2 r_1}$ present themselves at first as the squares of $+\dfrac{\sin d}{\sin r}$ and $+\dfrac{\sin d_1}{\sin r_1}$, and not of $-\dfrac{\sin d}{\sin r}$ and $-\dfrac{\sin d_1}{\sin r_1}$.

Cor.—When the radius of the sphere becomes infinite, the sphere is then a plane, and the above evidently becomes

$$\frac{d}{r}-\frac{d_1}{r_1}=2.$$

The following independent proof of the property when the circles are in a plane may not be devoid of interest to the student.

Let O, O_1, O_2, O_3 be the centres of the four circles in mutual contact on a plane; r, r_1, r_2, r_3 their respective radii; d and s the altitude and segment

of the base adjacent to O_3 of the triangle OO_2O_3, and d_1, s_1, the corresponding lines in the triangle $O_1O_2O_3$. Then it is readily shewn that

$$s=r\frac{r_2-r_3}{r_2+r_3}+r_3\ldots\ldots\ldots(1) \qquad s_1=r_1\frac{r_2-r_3}{r_2+r_3}+r_3\ldots\ldots\ldots(2)$$

$$d=\frac{2\sqrt{rr_2r_3(r+r_2+r_3)}}{r_2+r_3}\ldots\ldots(3) \qquad d_1=\frac{2\sqrt{r_1r_2r_3(r_1+r_2+r_3)}}{r_2+r_3}\ldots\ldots(4)$$

Also, $OO^2_1=(s-s_1)^2+(d-d_1)^2$, or $(r+r_1)^2=\dfrac{(r-r_1)^2(r_2-r_3)^2}{(r_2+r_3)^2}+(d-d_1)^2$;

$$\therefore \left(\frac{r_2-r_3}{r_2+r_3}\right)^2=\left(\frac{r_1+r}{r_1-r}\right)^2-\left(\frac{d_1-d}{r_1-r}\right)^2\ldots\ldots\ldots\ldots(5)$$

Again, $rd^2_1-r_1d^2=\dfrac{4rr_1r_2r_3(r_1-r)}{(r_2+r_3)^2}=rr_1(r_1-r)\left\{1-\left(\dfrac{r_2-r_3}{r_2+r_3}\right)^2\right\}$;

$$\therefore \left(\frac{r_2-r_3}{r_2+r_3}\right)^2=1-\frac{rd^2_1-r_1d^2}{rr_1(r_1-r)}=\left(\frac{r_1+r}{r_1-r}\right)^2-\left(\frac{d_1-d}{r_1-r}\right)^2, \text{ by (5).}$$

Hence, by reduction, we have

$$4r^2r^2_1=r^2_1d^2-2rr_1dd_1+r^2d^2_1;$$

$$\therefore\ r_1d-rd_1=\pm 2rr_1, \text{ or } \frac{d}{r}-\frac{d_1}{r_1}=\pm 2.$$

But it is easy to find that

$$\frac{d^2}{r^2}-\frac{d^2_1}{r^2_1}=\frac{4r_2r_3(r_1-r)}{rr_1(r_2+r_3)};$$

which is positive, and therefore $\dfrac{d}{r}-\dfrac{d_1}{r_1}$, which is equal to $\dfrac{d^2}{r^2}-\dfrac{d^2_1}{r^2_1}$ divided by $\dfrac{d}{r}+\dfrac{d_1}{r_1}$, is also positive; consequently we have finally

$$\frac{d}{r}-\frac{d_1}{r_1}=2.$$

XV.—*Mr. Philip Beecroft, Hyde, Cheshire.*

The sum of the squares of the cosecants of the perpendiculars from the angles of a spherical triangle upon the opposite sides, is equal to one-fourth of the sum of the squares of the cosecants of the radii of the four circles touching the sides of the triangle.

[FIRST SOLUTION.—*Mr. P. Beecroft, the Proposer.*]

Let p_1, p_2, p_3 denote the perpendiculars from the angles upon the sides; r, r_1, r_2, r_3 the radii of the inscribed circles; s the half sum of the sides a, b, c, and

$$n^2=\sin s\sin(s-a)\sin(s-b)\sin(s-c).$$

Then, by prop. *A*, pp. 100 and 101 of the Mathematician, we readily obtain

$$\text{cosec}^2r=\cot(s-a)\cot(s-b)+\cot(s-b)\cot(s-c)+\cot(s-c)\cot(s-a)\cdot$$
$$\text{cosec}^2r_1=\cot(s-b)\cot(s-c)-\cot s\cot(s-b)-\cot s\cot(s-c)$$
$$\text{cosec}^2r_2=\cot(s-a)\cot(s-c)-\cot s\cot(s-a)-\cot s\cot(s-c)$$
$$\text{cosec}^2r_3=\cot(s-a)\cot(s-b)-\cot s\cot(s-a)-\cot s\cot(s-b)$$

By addition, $\operatorname{cosec}^2 r + \operatorname{cosec}^2 r_1 + \operatorname{cosec}^2 r_2 + \operatorname{cosec}^2 r_3$

$$=2\left\{\begin{array}{l}\cot(s-a)\cot(s-b)+\cot(s-b)\cot(s-c)+\cot(s-c)\cot(s-a)\\ -\cot s\cot(s-a)-\cot s\cot(s-b)-\cot s\cot(s-c)\end{array}\right\}\ldots\ (1)$$

Again, $$4\operatorname{cosec}^2 p_1 = \frac{\sin^2 a}{n^2} = \frac{\sin\{s-(s-a)\}\sin\{(s-b)+(s-c)\}}{n^2}$$
$$= \{\cot(s-a)-\cot s\}\,\{\cot(s-b)+\cot(s-c)\};$$

Similarly, $$4\operatorname{cosec}^2 p_2 = \{\cot(s-b)-\cot s\}\,\{\cot(s-a)+\cot(s-c)\}$$
$$4\operatorname{cosec}^2 p_3 = \{\cot(s-c)-\cot s\}\,\{\cot(s-a)+\cot(s-b)\}.$$

Adding the last three equations, we get
$$4(\operatorname{cosec}^2 p_1 + \operatorname{cosec}^2 p_2 + \operatorname{cosec}^2 p_3)$$
$$=2\left\{\begin{array}{l}\cot(s-a)\cot(s-b)+\cot(s-b)\cot(s-c)+\cot(s-c)\cot(s-a)\\ -\cot s\cot(s-a)-\cot s\cot(s-b)-\cot s\cot(s-c)\end{array}\right\}\ldots.(2)$$

Hence, comparing (1) and (2), we get
$$4(\operatorname{cosec}^2 p_1+\operatorname{cosec}^2 p_2+\operatorname{cosec}^2 p_3) = \operatorname{cosec}^2 r+\operatorname{cosec}^2 r_1+\operatorname{cosec}^2 r_2+\operatorname{cosec}^2 r_3.$$

[SECOND SOLUTION.—*Mr. Samuel Bills, Hawton; and similarly by Mr. Anderson.*]

Lemma.—If a, b, c denote the three sides of a spherical triangle, and s half their sum; then
$$\sin^2 s+\sin^2(s-a)+\sin^2(s-b)+\sin^2(s-c)=2-2\cos a\cos b\cos c.$$

For by the usual formulæ we have
$$\sin^2 s=\tfrac{1}{2}\{1-\cos(a+b+c)\} = \tfrac{1}{2}-\tfrac{1}{2}\cos(a+b)\cos c + \tfrac{1}{2}\sin(a+b)\sin c$$

Similarly,
$$\sin^2(s-a)=\tfrac{1}{2}\{1-\cos(b+c-a)\} = \tfrac{1}{2}-\tfrac{1}{2}\cos(a-b)\cos c - \tfrac{1}{2}\sin(a-b)\sin c$$
$$\sin^2(s-b)=\tfrac{1}{2}\{1-\cos(a-b+c)\} = \tfrac{1}{2}-\tfrac{1}{2}\cos(a-b)\cos c + \tfrac{1}{2}\sin(a-b)\sin c$$
$$\sin^2(s-c)=\tfrac{1}{2}\{1-\cos(a+b-c)\} = \tfrac{1}{2}-\tfrac{1}{2}\cos(a+b)\cos c - \tfrac{1}{2}\sin(a+b)\sin c$$

Hence, by addition, we get
$$\sin^2 s+\sin^2(s-a)+\sin^2(s-b)+\sin^2(s-c)=2-\cos c\{\cos(a+b)+\cos(a-b)\}$$
$$=2-2\cos a\cos b\cos c.$$

Demonstration of the Proposition.—Let p_1, p_2, p_3 be the three perpendiculars from the angles on the opposite sides, and let also
$$4n^2 = 1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c\ldots\ldots\ldots\ldots(1)$$

Then by the properties of spherical triangles, we shall have
$$\sin p_1 = \sin c\sin B = \frac{2n}{\sin a}\ \therefore\ \operatorname{cosec}^2 p_1 = \frac{\sin^2 a}{4n^2} = \frac{1-\cos^2 a}{4n^2};$$
$$\sin p_2 = \sin a\sin C = \frac{2n}{\sin b}\ \therefore\ \operatorname{cosec}^2 p_2 = \frac{\sin^2 b}{4n^2} = \frac{1-\cos^2 b}{4n^2};$$
$$\sin p_3 = \sin b\sin A = \frac{2n}{\sin c}\ \therefore\ \operatorname{cosec}^2 p_3 = \frac{\sin^2 c}{4n^2} = \frac{1-\cos^2 c}{4n^2}$$
$$\therefore\ 4(\operatorname{cosec}^2 p_1+\operatorname{cosec}^2 p_2+\operatorname{cosec}^2 p_3) = \frac{3-\cos^2 a-\cos^2 b-\cos^2 c}{n^2}\ldots.(2)$$

Again, if r, r_1, r_2, r_3 denote the radii of the four inscribed circles, we have
$$\cot r = \frac{\sin s}{n},\ \text{and therefore}\ \operatorname{cosec}^2 r = \frac{n^2+\sin^2 s}{n^2};$$

Also, $\cot r_1 = \dfrac{\sin(s-a)}{n}$, ... $\operatorname{cosec}^2 r_1 = \dfrac{n^2+\sin^2(s-a)}{n^2}$

$\cot r_2 = \dfrac{\sin(s-b)}{n}$, ... $\operatorname{cosec}^2 r_2 = \dfrac{n^2+\sin^2(s-b)}{n^2}$

$\cot r_3 = \dfrac{\sin(s-c)}{n}$, ... $\operatorname{cosec}^2 r_3 = \dfrac{n^2+\sin^2(s-c)}{n^2}$.

Hence, by addition and the *lemma*, we have

$$\operatorname{cosec}^2 r + \operatorname{cosec}^2 r_1 + \operatorname{cosec}^2 r_2 + \operatorname{cosec}^2 r_3 = \frac{4n^2 + 2 - 2\cos a \cos b \cos c}{n^2}$$

$$= 4(\operatorname{cosec}^2 p_1 + \operatorname{cosec}^2 p_2 + \operatorname{cosec}^2 p_3), \text{ by } (1, 2).$$

[THIRD SOLUTION.—*Mr. Weddle.*]

Let $n^2 = \sin s \sin(s-a) \sin(s-b) \sin(s-c)$,
and $N^2 = -\cos S \cos(S-A) \cos(S-B) \cos(S-C)$.

Then Young's Trig., first edit., pp. 225 and 228,

$$\left.\begin{aligned} \tan r &= \frac{n}{\sin s}, \quad \tan r_2 = \frac{n}{\sin(s-b)} \\ \tan r_1 &= \frac{n}{\sin(s-a)}, \quad \tan r_3 = \frac{n}{\sin(s-c)} \end{aligned}\right\} \ldots\ldots\ldots\ldots (1)$$

$$\left.\begin{aligned} \text{and, } \cot R &= \frac{N}{-\cos S}, \quad \cot R_2 = \frac{N}{\cos(S-B)} \\ \cot R_1 &= \frac{N}{\cos(S-A)}, \quad \cot R_3 = \frac{N}{\cos(S-C)} \end{aligned}\right\} \ldots\ldots\ldots\ldots (2)$$

Now if p_1, p_2, p_3 be the perpendiculars on the sides a, b, c respectively, we evidently have,

$\sin p_1 = \sin b \sin C$, but $\sin b = \dfrac{2N}{\sin A \sin C}$, and $\sin C = \dfrac{2n}{\sin a \sin b}$, hence

$$\left.\begin{aligned} \sin p_1 &= \frac{2n}{\sin a} = \frac{2N}{\sin A} \\ \sin p_2 &= \frac{2n}{\sin b} = \frac{2N}{\sin B} \\ \sin p_3 &= \frac{2n}{\sin c} = \frac{2N}{\sin C} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Hence,

$$\sin p_1 \sin p_2 \sin p_3 = \frac{2n}{\sin a} \cdot \frac{2n}{\sin b} \cdot \frac{2N}{\sin C} = 4Nn, \text{ for } 2n = \sin a \sin b \sin C;$$

$$\therefore \quad \sin p_1 \sin p_2 \sin p_3 = 4nN = \sin a \sin b \sin c \sin A \sin B \sin C \ldots (4)$$

Again, from (1),

$$\cot^2 r = \frac{\sin s}{\sin(s-a)\sin(s-b)\sin(s-c)} = \frac{\sin\{(s-a)+(s-b)+(s-c)\}}{\sin(s-a)\sin(s-b)\sin(s-c)}$$

$$= \cot(s-a)\cot(s-b) + \cot(s-a)\cot(s-c) + \cot(s-b)\cot(s-c) - 1.$$

Hence,

$$\left.\begin{aligned}
\operatorname{cosec}^2 r &= \cot(s-a)\cot(s-b)+\cot(s-a)\cot(s-c)+\cot(s-b)\cot(s-c)\\
\operatorname{cosec}^2 r_1 &= -\cot s.\cot(s-b)-\cot s.\cot(s-c)+\cot(s-b)\cot(s-c)\\
\operatorname{cosec}^2 r_2 &= -\cot s.\cot(s-a)-\cot s.\cot(s-c)+\cot(s-a)\cot(s-c)\\
\operatorname{cosec}^2 r_3 &= -\cot s.\cot(s-a)-\cot s.\cot(s-b)+\cot(s-a)\cot(s-b)
\end{aligned}\right\}\ldots(5)$$

Similarly, from (2),

$$\begin{aligned}
\sec^2 R &= \tan(S-A)\tan(S-B)+\tan(S-A)\tan(S-C)+\tan(S-B)\tan(S-C)\\
\sec^2 R_1 &= -\tan S.\tan(S-B)-\tan S.\tan(S-C)+\tan(S-B)\tan(S-C)\\
\sec^2 R_2 &= -\tan S.\tan(S-A)-\tan S.\tan(S-C)+\tan(S-A)\tan(S-C)\\
\sec^2 R_3 &= -\tan S.\tan(S-A)-\tan S.\tan(S-B)+\tan(S-A)\tan(S-B)
\end{aligned}\quad\ldots.(6)$$

Again, $\sin a=\sin\{s-(s-a)\}=\sin s\cos(s-a)-\cos s\sin(s-a)$
$=\sin s\sin(s-a)\{\cot(s-a)-\cot s\}$,

$\sin a=\sin\{(s-b)+(s-c)\}=\sin(s-b)\sin(s-c)\{\cot(s-b)+\cot(s-c)\}$.

Multiply these two values of $\sin a$, divide by n^2, therefore (3)

$$4\operatorname{cosec}^2 p_1=\{\cot(s-a)-\cot s\}\{\cot(s-b)+\cot(s-c)\}.$$

In a similar manner, we get

$$4\operatorname{cosec}^2 p_1=\{\tan(S-A)-\tan S\}\{\tan(S-B)+\tan(S-C)\}.$$

$$\left.\begin{aligned}
\therefore\ 4\operatorname{cosec}^2 p_1 &= \{\cot(s-a)-\cot s\}\{\cot(s-b)+\cot(s-c)\}\\
&= \{\tan(S-A)-\tan S\}\{\tan(S-B)+\tan(S-C)\}\\
4\operatorname{cosec}^2 p_2 &= \{\cot(s-b)-\cot s\}\{\cot(s-a)+\cot(s-c)\}\\
&= \{\tan(S-B)-\tan S\}\{\tan(S-A)+\tan(S-C)\}\\
4\operatorname{cosec}^2 p_3 &= \{\cot(s-c)-\cot s\}\{\cot(s-a)+\cot(s-b)\}\\
&= \{\tan(S-C)-\tan S\}\{\tan(S-A)+\tan(S-B)\}
\end{aligned}\right\}\ldots.(7)$$

By adding the equations in (5), (6), and (7), we have

$$\begin{aligned}
\operatorname{cosec}^2 r+\operatorname{cosec}^2 r_1+\operatorname{cosec}^2 r_2+\operatorname{cosec}^2 r_3 &= \sec^2 R+\sec^2 R_1+\sec^2 R_2+\sec^2 R_3\\
&= 4\operatorname{cosec}^2 p_1+4\operatorname{cosec}^2 p_2+4\operatorname{cosec}^2 p_3\ldots(8)
\end{aligned}$$

(8) establishes the theorem announced. I shall add one or two theorems intimately connected with the above.

Since,

$$\begin{aligned}
\sin s\sin(s-a)+\sin(s-b)\sin(s-c) &= \tfrac{1}{2}\cos a-\tfrac{1}{2}\cos(b+c)+\tfrac{1}{2}\cos(b-c)-\tfrac{1}{2}\cos a\\
&= \sin b\sin c.
\end{aligned}$$

Divide by n^2

$$\begin{aligned}
\therefore\ \operatorname{cosec} s\operatorname{cosec}(s-a)+\operatorname{cosec}(s-b)\operatorname{cosec}(s-c) &= \frac{\sin b\sin c}{n^2}\\
&= 4\operatorname{cosec} p_2\operatorname{cosec} p_3,\ \text{by (3)}.
\end{aligned}$$

Similarly, $4\operatorname{cosec} p_2\operatorname{cosec} p_3=-\sec S\sec(S-A)+\sec(S-B)\sec(S-C)$.

Hence,

$$\left.\begin{aligned}
4\operatorname{cosec} p_2\operatorname{cosec} p_3 &= \operatorname{cosec} s\operatorname{cosec}(s-a)+\operatorname{cosec}(s-b)\operatorname{cosec}(s-c)\\
&= -\sec S\sec(S-A)+\sec(S-B)\sec(S-C)\\
4\operatorname{cosec} p_1\operatorname{cosec} p_3 &= \operatorname{cosec} s\operatorname{cosec}(s-b)+\operatorname{cosec}(s-a)\operatorname{cosec}(s-c)\\
&= -\sec S\sec(S-B)+\sec(S-A)\sec(S-C)\\
4\operatorname{cosec} p_1\operatorname{cosec} p_2 &= \operatorname{cosec} s\operatorname{cosec}(s-c)+\operatorname{cosec}(s-a)\operatorname{cosec}(s-b)\\
&= -\sec S\sec(S-C)+\sec(S-A)\sec(S-B)
\end{aligned}\right\}\ldots.(9)$$

By (3), $\sin p_2 \sin p_3 = \dfrac{4n^2}{\sin b \sin c} = \dfrac{4N^2}{\sin B \sin C}$, but $\sin b \sin c = \dfrac{2n}{\sin A}$,

and $\sin B \sin C = \dfrac{2N}{\sin a}$;

$$\left.\begin{aligned} \therefore\ \sin p_2 \sin p_3 &= 2n \sin A = 2N \sin a \\ \sin p_1 \sin p_3 &= 2n \sin B = 2N \sin b \\ \sin p_1 \sin p_2 &= 2n \sin C = 2N \sin c \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (10)$$

Again, by a well known formula, $\sin^2 \frac{1}{2}A = \dfrac{\sin(s-b)\sin(s-c)}{\sin b \sin c}$, but, (1)

$\tan r_2 \tan r_3 = \dfrac{n^2}{\sin(s-b)\sin(s-c)}$, and $\sin p_2 \sin p_3 = \dfrac{4n^2}{\sin b \sin c}$, by (3);

$$\left.\begin{aligned} \therefore\ 4\sin^2 \tfrac{1}{2}A &= \cot r_2 \cot r_3 \sin p_2 \sin p_3 \\ 4\sin^2 \tfrac{1}{2}B &= \cot r_1 \cot r_3 \sin p_1 \sin p_3 \\ 4\sin^2 \tfrac{1}{2}C &= \cot r_1 \cot r_2 \sin p_1 \sin p_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (11)$$

Similarly,

$$\left.\begin{aligned} 4\cos^2 \tfrac{1}{2}A &= \cot r \cot r_1 \sin p_2 \sin p_3 \\ 4\cos^2 \tfrac{1}{2}B &= \cot r \cot r_2 \sin p_1 \sin p_3 \\ 4\cos^2 \tfrac{1}{2}C &= \cot r \cot r_3 \sin p_1 \sin p_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (12)$$

$$\left.\begin{aligned} 4\sin^2 \tfrac{1}{2} a &= \tan R \tan R_1 \sin p_2 \sin p_3 \\ 4\sin^2 \tfrac{1}{2} b &= \tan R \tan R_2 \sin p_1 \sin p_3 \\ 4\sin^2 \tfrac{1}{2} c &= \tan R \tan R_3 \sin p_1 \sin p_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (13)$$

And

$$\left.\begin{aligned} 4\cos^2 \tfrac{1}{2} a &= \tan R_2 \tan R_3 \sin p_2 \sin p_3 \\ 4\cos^2 \tfrac{1}{2} b &= \tan R_1 \tan R_3 \sin p_1 \sin p_3 \\ 4\cos^2 \tfrac{1}{2} c &= \tan R_1 \tan R_2 \sin p_1 \sin p_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (14)$$

Add the equations in (12) severally to those in (11), and divide by $\sin p_2 \sin p_3$, $\sin p_1 \sin p_3$ and $\sin p_1 \sin p_2$ respectively; and similarly with (13) and (14): hence we obtain

$$\left.\begin{aligned} 4\operatorname{cosec} p_2 \operatorname{cosec} p_3 &= \cot r \cot r_1 + \cot r_2 \cot r_3 = \tan R \tan R_1 + \tan R_2 \tan R_3 \\ 4\operatorname{cosec} p_1 \operatorname{cosec} p_3 &= \cot r \cot r_2 + \cot r_1 \cot r_3 = \tan R \tan R_2 + \tan R_1 \tan R_3 \\ 4\operatorname{cosec} p_1 \operatorname{cosec} p_2 &= \cot r \cot r_3 + \cot r_1 \cot r_2 = \tan R \tan R_3 + \tan R_1 \tan R_2 \end{aligned}\right\} (15)$$

$$\therefore\ 4\operatorname{cosec} p_1 \operatorname{cosec} p_2 + 4\operatorname{cosec} p_1 \operatorname{cosec} p_3 + 4\operatorname{cosec} p_2 \operatorname{cosec} p_3$$
$$= \cot r \cot r_1 + \cot r \cot r_2 + \cot r \cot r_3 + \cot r_1 \cot r_2 + \cot r_1 \cot r_3 + \cot r_2 \cot r_3$$
$$= \tan R \tan R_1 + \tan R \tan R_2 + \tan R \tan R_3 + \tan R_1 \tan R_2 + \tan R_1 \tan R_3 + \tan R_2 \tan R_3 \ldots (16)$$

Write $1+\cot^2$ for cosec^2 in the first member of (8), and $1+\tan^2$ for $\sec^2$ in the second: to (8), so modified, add twice (16), and we shall find

$$4\{\operatorname{cosec} p_1 + \operatorname{cosec} p_2 + \operatorname{cosec} p_3\}^2 = \{\cot r + \cot r_1 + \cot r_2 + \cot r_3\}^2 + 4$$
$$= \{\tan R + \tan R_1 + \tan R_2 + \tan R_3\}^2 + 4.$$

Again to (8) add twice the value of $4\operatorname{cosec} p_2 \operatorname{cosec} p_3 - 4\operatorname{cosec} p_1 \operatorname{cosec} p_3 - 4\operatorname{cosec} p_1 \operatorname{cosec} p_2$ derived from (15), and we have

$$4\{-\operatorname{cosec} p_1 + \operatorname{cosec} p_2 + \operatorname{cosec} p_3\}^2 = \{-\cot r - \cot r_1 + \cot r_2 + \cot r_3\}^2 + 4$$
$$= \{\tan R + \tan R_1 - \tan R_2 - \tan R_3\}^2 + 4.$$

Collecting the whole, we thus have

$$\left.\begin{aligned}
4\{\operatorname{cosec} p_1+\operatorname{cosec} p_2+\operatorname{cosec} p_3\}^2&=\{\cot r+\cot r_1+\cot r_2+\cot r_3\}^2+4\\
&=\{\tan R+\tan R_1+\tan R_2+\tan R_3\}^2+4\\
4\{-\operatorname{cosec} p_1+\operatorname{cosec} p_2+\operatorname{cosec} p_3\}^2&=\{-\cot r-\cot r_1+\cot r_2+\cot r_3\}^2+4\\
&=\{\tan R+\tan R_1-\tan R_2+\tan R_3\}^2+4\\
4\{\operatorname{cosec} p_1-\operatorname{cosec} p_2+\operatorname{cosec} p_3\}^2&=\{-\cot r+\cot r_1-\cot r_2+\cot r_3\}^2+4\\
&=\{\tan R-\tan R_1+\tan R_2-\tan R_3\}^2+4\\
4\{\operatorname{cosec} p_1+\operatorname{cosec} p_2-\operatorname{cosec} p_3\}^2&=\{-\cot r+\cot r_1+\cot r_2-\cot r_3\}^2+4\\
&=\{\tan R-\tan R_1-\tan R_2+\tan R_3\}^2+4
\end{aligned}\right\}(17)$$

The right hand members of (17) admit of transformations by means of equations (B), No. 1, p. 20. Thus

$$\cot r+\cot r_1+\cot r_2+\cot r_3=2(\tan R+\cot r)=2(\tan R_1+\cot r_1)=etc.$$
$$-\cot r-\cot r_1+\cot r_2+\cot r_3=2(\tan R-\cot r_1)=2(\tan R_1-\cot r)=etc.$$

But it is perhaps unnecessary to do more than to hint at these modifications

MATHEMATICAL EXERCISES—(*continued.*)

16.—*James Lockhart, Esq.*

Find the relation between the roots of the two equations

$$x^3-bx=c\ldots.(1)\qquad x^3+bx^2=c^2\ldots.(2)$$

17.—*Mr. Rutherford.*

Eliminate θ from the equations

$$m\tan 2\theta-n\tan 2\phi=0\ldots\ldots\ldots\ldots\ldots..(1)$$
$$m\cot\phi-n\cot\theta=p\operatorname{cosec}^2\theta\ldots\ldots\ldots.(2)$$

and give the resulting equation in terms of ϕ, when

$$m=W'b+Wa,\quad n=W'b-Wa,\quad p=(W+W')r.$$

18.—*Mr. St. Andrew St. John, Gent. Cadet, R. M. Academy.*

Trace the curve whose equation is

$$y^2=\frac{x^3-b^3}{x-a}.$$

19.—*Mr. Thomas Weddle, Newcastle.*

Required the magnitude and position of the circles touching two sides of triangle and the circumscribing circle.

20.—*Mr. R. H. Wright, London.*

In an inverted cycloid, if δ represent the difference of the times of asce and descent of each complete oscillation of a body moving in it in a mediu whose resistance $=2kv$; show that

$$\delta=\frac{2l}{\sqrt{g-lk^2}}\sin^{-1}k\sqrt{\frac{l}{g}}.$$

21.—*Mr. James Dalmahoy, Edinburgh.*

Let a tangent be drawn to a conic section at either of its vertices, a likewise its circle of curvature at the same point; then any line drawn touch one of the curves and to cut the other and the tangent, will be harn nically divided in the points of contact and intersection.

22.—*Mr. Philip Beecroft, Hyde, Cheshire.*

Let O, O_1, O_2, O_3 be the centres of four circles in mutual contact on a plane, and P, P_1, P_2, P_3 the centres of the circles touching each other mutually in the same points as the other four. Join the centres O_1, O_2, O_3, and also the centres P_1, P_2, P_3. Let U be the intersection of the perpendiculars from the angles of the triangle $O_1O_2O_3$ upon its opposite sides, and V the intersection of the perpendiculars from the angles of the triangle $P_1P_2P_3$ upon its opposite sides; also let O_4 be the centre of another circle touching the three circles centres O_1, O_2, O_3; and P_4 the centre of another circle touching the three circles centres P_1, P_2, P_3; then the *six points* O, P, O_4, P_4, U, V will all range in a right line.

23.—*Mr. W. S. B. Woolhouse, Editor of the Lady's and Gentleman's Diary.*

According to the usual functional notation

$\sin^2 x$ denoting sin. of $\sin x$
$\sin^3 x$,, sin. of $\sin^2 x$
....

it is required to find the limiting value of the fraction

$$\frac{x^{n-1}\sin^n x - (\sin x)^n}{x^6}$$

when $x = 0$.

24.—*Mr. Geo. W. Hearn, R. M. Coll., Sandhurst.*

Let there be three circles in the same plane, then in general *eight* other circles may be drawn touching all the three; let r_n be the radius of one which is touched *externally* by one of the given circles, and $\Sigma \frac{1}{r_n}$ = the sum of the reciprocals of the radii of all the circles which are touched externally by n of the given circles; then

$$\frac{1}{r_3} + \Sigma\frac{1}{r_1} = \frac{1}{r_0} + \Sigma\frac{1}{r_2}.$$

*** We have particularly to request those correspondents, who favor us with new questions, to accompany them with at least a sketch of the intended solutions. A problem may be impossible, an alleged theorem not true: and it is not always convenient to the editors to give the time and attention requisite for investigating the question, *ab initio*, between the period at which it is ordinarily received and required for use.

NOTE ON PROBABILITY.

(From a Correspondent.)

In Mr. Babbage's valuable and very original work, called The Ninth Bridgewater Treatise, there is a mathematical argument in reference to Hume's celebrated Essay on Miracles, the conclusion of which is that twenty-five witnesses, who bear independent testimony to the occurrence of a miracle, are sufficient to establish the fact against the *à priori* improbability of the event, as inferred from the uniform course of nature; provided the veracity of the witnesses is such, that each witness makes but one false statement to ten true ones. There appears to be an imperfection in a certain

part of this argument; in consequence of which the number of witnesses necessary to produce *odds* in favour of the event is greatly increased.

The probability of falsehood for each of the n witnesses being $\frac{1}{p}$, and the *à priori* probability of the non-occurrence of the event being $\frac{1}{m+2}$, Mr. Babbage correctly infers (p. 243,) that the probability *for* the event—both considerations combined—is

$$\frac{(p-1)^n}{(p-1)^n+m+1},$$

and the probability *against* it, or in favour of its non-occurrence,

$$\frac{m+1}{(p-1)^n+m+1}.$$

Now it is sufficient, to preclude a reasonable rejection of the fact affirmed, or to render the *odds* in its favour, that

$$(p-1)^n > m+1$$

$$\text{or, } n\log(p-1) > \log(m+1)$$

$$\therefore n > \frac{\log(m+1)}{\log(p-1)}.$$

By hypothesis $p = 11$, and $m+1 = 10^{12}$,

$$\therefore n > \frac{12}{1} > 12.$$

Consequently thirteen such witnesses are sufficient to render the occurrence more probable than improbable.

Belfast, May 15, 1844. Y.

ON THE FORMATION OF NUMERICAL EQUATIONS HAVING NEARLY EQUAL ROOTS.

It might be supposed that a numerical equation of any degree could be readily formed, having any number of roots nearly equal to each other, by first forming an equation, having the same number of equal roots, and then making a slight numerical change, either in the absolute term of that equation, or in one or more of its coefficients. Trivial, however, as any modification of the co-efficients, or of the absolute term of an equation may be, it will be found that even a small change in this respect will give rise to an equation whose roots, instead of being nearly equal, may not only differ very considerably from the roots of the original equation, but any number of pairs of them may even be imaginary. Thus for example, the cubic equation

$$x^3-15x^2+75x-125=0$$

has three roots each equal to 5. Now the product of all the roots of this equation is numerically equal to 125, and by slightly altering this product, we should imagine that the equation so modified would have its roots nearly equal to each other, and that such a trivial numerical change in the *product* of all the roots would have but little tendency, either to increase or decrease

the roots, or to introduce a pair of imaginary ones. Such is the fact, however, that if we take the equation

$$x^3 - 15x^2 + 75x - 124 = 0,$$

it will be found to have only one real root equal to 4, and the remaining roots imaginary. This unexpected result is very remarkable, and similar results follow from other modifications, showing that the minutest change of the co-efficients may produce an equation whose roots have a character very different from that of the roots of the original equation; and hence that some more certain mode of forming such equations must be adopted.

The following method, which was communicated to us in 1842, by Mr. James Lockhart, the venerable author of several interesting tracts on the resolution of equations, appears to be well adapted for the formation of such equations.

Find two or more integer numbers whose square roots have several figures in the decimal part common to all the roots; increase or decrease the integer parts of these roots by such arbitrary numbers as will make the roots coincide, both in integers and decimals, to the same extent; then as surd roots occur in pairs in any equation, the employment of the double sign before the surd part will furnish us with the several roots of an equation, which may be formed in the usual manner. Thus

$$\sqrt{5} = 2{\cdot}236068 \text{ and } \sqrt{451} = 21.2367606;$$

therefore we have the following system of equations,

$$\begin{aligned} \sqrt{5} &= -19 + \sqrt{451} \\ -1 + \sqrt{5} &= -20 + \sqrt{451} \\ 1 + \sqrt{5} &= -18 + \sqrt{451} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and hence taking the members on each side of any of these equations for two of the roots, the other corresponding roots are also known. Taking the second of these equations, we have then the four roots

$$-1 + \sqrt{5}, \quad -1 - \sqrt{5}, \quad -20 + \sqrt{451}, \quad -20 - \sqrt{451};$$

and the equation of which these numbers are the roots, is

$$x^4 + 42x^3 + 25x^2 - 262x + 204 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

This equation has, therefore, two roots nearly equal to each other, one being 1·236068, and the other 1·2367606, agreeing as far as the third place of decimals inclusive; and by increasing or diminishing the roots, we may form equations at pleasure, and select for practice those which have the smallest co-efficients.

By multiplying (1) by $x \pm a$, where a may be any integer number whatever, we should obtain an equation of the fifth degree having two roots nearly equal, and by taking another integer number whose square root coincides to the same extent in the decimal part with the former two, we may form an equation of the sixth degree having three roots nearly equal to each other, and so on to any extent.

The three following equations were formed by Mr. Lockhart, and proposed to British students for resolution in 1841:

$$x^5 - 82x^4 + 2404x^3 - 26394x^2 + 6132x - 360 = 0$$

$$x^5 + 173x^4 + 2356x^3 + 10468x^2 - 14101x + 4183 = 0$$

$$x^6 + 378x^5 + 38189x^4 + 492368x^3 - 572554x^2 + 213720x - 26352 = 0.$$

Several of the roots of each of these equations coincide to six decimal places.

ON THE THEORY OF CO-ORDINATES.

[*Mr. Thomas Dobson, Totteridge, Herts.*]

By the aid of co-ordinates and certain conventional meanings attached to algebraical symbols, the Cartesian Geometry enables us to draw, as it were, a geometrical picture of the several states of an algebraical function corresponding to the successive changes of value of the variables of which it is composed; and, conversely, by the same fertile method may be exhibited a faithful analytical representation of a given geometrical figure.

Among the numerous systems of co-ordinates which may be employed in the performance of these operations, it is important to distinguish, on the one hand, those which offer the greatest facilities for depicting equations geometrically; and on the other, those which produce the analytical representation of the given figure in its simplest form, or in that form from which its properties may be most conveniently evolved, as may be required.

Since the determination of geometrical magnitude may generally be effected by finding its projections on certain planes, it will suffice to consider here plane curves and their equations.

Continuous plane curves may either be traced by joining a considerable number of approximate points, or described mechanically by continued motion. The first process will be most easily executed when the co-ordinates to be measured in order to determine each point have been selected from the first elements of figure, as straight lines, angles, *etc.*; and the proposed object will be best accomplished in the second instance when the variation of the position of the co-ordinates is produced by the simplest motions, viz. those of translation or rotation.

Of the various possible combinations of the elements of figure and motion, a little consideration will shew that the system best adapted for graphical purposes, both on account of the simplicity of the co-ordinates themselves, as of their mode of variation, is that by which the curve is generated by the intersection of two right lines having motions of translation parallel to two fixed axes.

Another very useful system is that which determines a curve by the translation of a point in a right line revolving about a fixed point or pole.

These two systems, known by the names of Rectilinear and Polar Co-ordinates, present such superior facilities for the graphical formation of curves, by reason of the simplicity of their elements, as to be almost exclusively preferred to all others by geometers, and are therefore the only ones treated of in our elementary works on Analytical Geometry. Nevertheless, in order fully to appreciate the justice of the preference which has hitherto been universally accorded to them, as well as to acquire wider views of the general theory of co-ordinates, the student would do well to attempt to trace a few equations through the medium of other systems, as for example, those which determine the curve by the intersection of two right lines revolving about two fixed points, or by the intersection of a revolving line with a line having a motion of translation.

The pictorial form of an equation will manifestly vary according to the system of co-ordinates employed in its formation, thus the equation $x = a$ will give a point or a circle, and $y = ax$ a straight line or a spiral, according as they are interpreted by means of rectilinear or polar co-ordinates; and if the distances of a point from a given right line and a fixed point be used,

the equation $x = my$ will produce an ellipse, hyperbola, or parabola, according as we take $m < 1$, $m > 1$, or $m = 1$. In like manner $x = a$ may characterize the conchoid or cissoid.

Now if we wish to perform the converse operation, and undertake to make an analytical representation of a geometrical plane figure given by definition, it will be found that no one system of co-ordinates recommends itself to our notice as being generally preferable to the others for this purpose. The process however will be greatly facilitated by observing that every definition of a curve which indicates, remotely or directly, a method of description either by points or continued motion, constitutes in itself a first equation to the curve relative to a certain system of co-ordinates virtually implied in the definition.

Thus, the circle being defined as the curve of which every point is equally distant from a fixed point, the definition itself furnishes spontaneously the equation $u = r$ in polar co-ordinates; and the definition of the locus of a point, the sum or difference of whose distances from two given points is constant, obviously suggests $d \pm d' = a$ as the primitive equations of the ellipse and hyperbola in this system.

In proceeding, therefore, to form the equation to a curve according to any proposed system of co-ordinates, the most natural and methodical course seems to be, first to write down the equation which is implicitly contained in the given definition, relative to the system of co-ordinates therein indicated, and afterwards to eliminate these by means of two relations between them and the proposed co-ordinates.

Different definitions of the same curve afford primitive equations which vary greatly as to facility of expression; thus, the elementary definition of a circle gives $u = r$, and $x^2 + y^2 = r^2$, in polar and rectilinear co-ordinates; and if it be defined as the locus of the intersection of two right lines inclined at an invariable angle a, and passing through two fixed points, it may be characterized by $\beta - \beta' = a$; the co-ordinates β, β' denoting the angles which the lines make with the line joining the fixed points. But, when the circle is defined as that curve which, of all those of equal length, contains the greatest area, the highest powers of analysis are required to form its equation.

In employing the ordinary system of polar co-ordinates to depict an equation geometrically, some mathematicians (vide M. Comte's Géométrie Analytique, Paris, 1843,) reject all negative values of the radius vector, on the ground that, occupying successively every direction in the plane of reference, it is not susceptible of opposite symbolical affections. Others, however, (vide De Morgan's Diff. and Integ. Calculus, page 342; and Gregory's Examples of the Processes of the Calculus, page 181,) observing that this restricted mode of interpreting polar equations often deprives curves of branches which they are found to possess when drawn by means of other systems of co-ordinates, contend, notwithstanding the apparent anomaly above-mentioned, that the negative values should be measured backwards, or, in a direction contrary to that of the radius vector. The completeness of figure which is always obtained by this method is a strong presumptive proof of its validity; and it is conceived the following considerations will tend to show that such a procedure is quite consistent with, and even a necessary consequence of, the fundamental principle of the Geometry of Descartes; viz. that every direct inversion of concrete magnitude may be characterized analytically by a corresponding change of sign. It is obvious that the fixed line and variable angle of this system, however

serviceable they may be found in the practical depicting of curves, are by no means necessary elements of the present question, which will remain unchanged if we suppose the curve to be generated by the motion of a point (P) in an invariable line (AB) of indefinite length, either right or curved, revolving about a fixed point (O). For, by the very nature of polar co-ordinates, the generating point P may be assumed to be always in AB, and no hypothesis has been introduced by this modification respecting the absolute motion of P, which may be any whatever, being compounded of the arbitrary motion of rotation of AB, and its own motion of translation in AB, which is dependent upon the law of the curve.

Now, suppose P to begin to move from any point C and in the direction AB, and let every part of the line described by P in that direction be considered positive; then, according to the general principle before mentioned, every part of the line described by P in the contrary direction BA must be considered negative. Consequently, whenever P is found in CA, the distance CP must be affected with a negative sign. For, here as in the rectilinear system, as soon as the motion of P becomes retrograde, which we shall suppose to occur at B, negative magnitude begins to be generated; and when the negative magnitude exceeds the positive, that is when P is in CA, the whole space described $CB - BP = -CP$ is negative. And this is true whatever may be the position of AB in its plane; the rotation of AB about O having just as little effect upon the sign of CP as the revolution of a sheet of cardboard about a pin stuck through the origin would have upon the signs of the rectilinear co-ordinates engaged in tracing out a curve upon its surface. Hence it appears that the sign of CP is in nowise affected by the position of AB relative to any fixed line through O, but depends solely upon the position of P with respect to the point C, whence its motion commenced.

This important application of the principle of Descartes affords a direct and satisfactory explanation of the changes of sign experienced by the trigonometrical secant, which has hitherto formed a marked desideratum in treatises on trigonometry.

In this case, P is the intersection of the indefinite right line AB with a geometrical tangent CD to the circle (the fig. is easily conceived.)

Let OC be perpendicular to CD, and suppose OB to coincide with OC at the beginning of the motion, the direction AB being as before that of positive magnitude. Then, as OB revolves through the first quadrant, OP increases in the positive direction AB; but when OB reaches the second quadrant, P is found in OA, and being measured in the direction BA, OP is negative. The same occurs in the third quadrant; but as OB traverses the fourth quadrant, P, falling between O and B, OP is again negative. Here, then, the very changes of sign which are known from other sources to characterize the secant, have been deduced as consequences of the principle just considered. In fact, it is clear that the secant is nothing but the radius vector to a right line, for in the general polar equation $r = \dfrac{P \sin a}{\sin (a - \theta)}$ to the right line, let $P = OC = 1$, and $a = 90°$, then $r = \dfrac{1}{\cos \theta} = \sec \theta$.

Therefore, in order to be consistent, those who reject negative radii vectores should discard also negative values of the trigonometrical secant.

March 15, 1844.

MODERN GEOMETRY.

[*Mr. Davies.*]

Il est toujours utile de remonter à l'origine des vérités géométriques, pour découvrir, de ce point de vue éléve, les différentes formes dont elles sont susceptibles et qui peuvent en étendre les applications.—*Chasles.*

The Mathematical Collections of Pappus, from their incomplete and even fragmentary character, disappoint our inquiries as to the real extent to which almost every subject treated in them had been carried by the Greek mathematicians; yet they, still, give us considerable insight into the objects, views, and modes of research, which prevailed amongst that unequalled race of Geometers. It is, however, owing to the extraordinary perseverance and sagacity of the moderns that we owe even any distinct conception of the true character of the treatises of Euclid and Apollonius: and it has probably required a greater demand upon mathematical skill to restore a few of the twenty-four books enumerated by Pappus to their primitive state, than was employed in the original composition of the entire series.

It also appears that these books, themselves, were only considered to be *subsidiary* to the ulterior and more difficult researches in which the geometers of Greece employed themselves:—in fact, as the solutions of those propositions in which it was found that their more recondite researches generally terminated.

This circumstance justifies our belief that we have, in our time, little, if any, notion of the real difficulty and extent of the subjects discussed by the ancient geometers. For it must, obviously, imply that whatever naturally and easily follows from the properties and constructions in these works (the conics of Apollonius was one of the twenty-four) must have been familiarly known in the school of Plato for several centuries. It is to be kept in mind, too, that the seventh book of Pappus is wholly composed of *lemmas* which were used in those particular treatises, and which had been considered in the brighter days of the academy as known to all—and known so well as to be assumed without proof, or even reference, in the works to which they were made subservient.*

Under such circumstances, too, it appears perfectly justifiable to infer that all the properties—and especially systems of properties leading to general processes of investigation—which flow from those lemmas were known in the school of Alexandria: for it was not the character of the Greek intellect to allow the obvious and natural consequences of a proposition to pass undiscussed, or discussed in a merely slight and casual manner. Nor was it the habit of the Greek mathematicians to content themselves with the mere discussion and determination of the several cases of a proposition without examining its proper place in the range of geometrical truth; nor yet to leave its logical character undetermined, or its peculiarities (if such it had) unclassified with respect to the general principles upon which the entire science was founded.

Now it is unquestionable that the germs of several parts of what Chasles emphatically terms *la Géométrie Récente,* are to be found, or are implied,

* Dr. Trail's Life of Simson, p. 155.

in the mathematical collections of Pappus and other works which are come down, in a more or less perfect state, to our time. Of these it may be sufficient here to specify—the *Method of Transversals*, *Reciprocal Polars*, the *Anharmonic Ratio*, and the *Section of a line in Involution*. Yet it is only very recently that these systems have been in any degree matured amongst the moderns—if, indeed, the term "matured" be even approximately correct, when applied to the present subjects. In this country, indeed, they may, generally speaking, be said to be unknown; as we believe that till a very recent period, the terms themselves were not to be found in any English work. Our object, then, in these papers will be to give a tolerably comprehensive discussion of these and certain collateral subjects, bringing the treatment of each of them down to include the more recent researches of the Continental Geometers, with such additions as we may be enabled to make to the general stock.

In some cases we may depart from the modes of investigation, as well as the order of classification of our predecessors: yet we shall studiously avoid all unnecessary or capricious departure from the models furnished by the distinguished geometers who have labored in this field—resting satisfied if we shall succeed in attracting to these beautiful and effective methods, the attention of the readers of our work. It must also be distinctly understood that we wish to be considered in the main, and in reference to these subjects especially, rather as expositors of the methods than as original discoverers—and especially so in the earlier stages of the series.

For such properties as have already been given in Hutton's Course, we must refer to the work itself: it being unnecessary to reprint extracts from a work which is, probably, in the hands of all our readers.

August 1, 1844.

CHAPTER I.

ANHARMONIC RATIO.

I.

The following is Commandine's version of prop. 129, book vii. of the Mathematical Collections of Pappus:—

In tres lineas AB, CA, AD ducantur duæ rectæ linæ HE, HD. Dico ut rectangulum, quod contenetur HE, GF ad contentum HG, FE, ita esse rectangulum HB, DC ad contentum HD, BC.

Now if we suppose the point H to be in a line also passing through A, we shall have precisely the fundamental property upon which Chasles has reared his beautiful system of *anharmonic ratios*. The name itself has been given in analogy to the particular case in which one of the above-named ratios is that of equality, and which is familiarly known to modern geometers as the *harmonic section of a line*.

Two demonstrations of the property are given in the second volume of Hutton's Course, pp. 239, 240: both different from that of Pappus—though that one is remarkably elegant. Chasles offers no proof of it, contenting himself with a reference to the Mathematical Collections.

The *form*, however, in which M. Chasles writes his anharmonic ratio is different from that of Pappus, and perhaps for the purposes of general application, rather more convenient: and as, in following out his inves-

tigations, it is probable that other writers will use his notation, it appears desirable to employ it here.

If any four points a, b, c, d be taken in a straight line; and if any two be considered as primarily conjugate to each other, and the other two secondarily conjugate to each other: then there will be three coincident forms of the anharmonic ratio, any one of which implies the other two: viz.

$\frac{ac}{ad} : \frac{bc}{bd}$ when a, b are primary conjugates,

$\frac{ab}{ad} : \frac{cb}{cd}$ .... a, c ....,

$\frac{ac}{ab} : \frac{dc}{db}$ .... a, d ....

There can be no other combination different in fact from these: for though twelve pairs of combinations in twos and twos may be made, they will be effectively but repetitions of these three. This will be apparent from the following arrangement:—

(1) ab, cd	(3) bc, ad	(1) cd, ab	(3) da, bc
(2) ac, bd	(2) bd, ac	(2) ca, db	(2) db, ca
(3) ad, bc	(1) ba, cd	(3) cb, ad	(1) dc, ab

in which the two primary conjugate points are written in juxtaposition, and the two secondary in juxtaposition, with a comma between the pairs: and it will be obvious that all those marked (1) are alike, all those marked (2) are alike, and all those marked (3) are alike.

Moreover, as I have pointed out in my general demonstration of the fundamental property, these three apparently separate cases are but cases of the construction of the figure as to the relative position of its component radiants, sa, sb, sc, sd; resembling in this respect the three forms of the simple quadrilateral *, and some other instances which might be quoted, in the theory of transversals.

II.

The theorem of Pappus shews that if two transversals be cut by four radiants in a, b, c, d and a', b', c', d', we shall have

$$\frac{ac}{ad} : \frac{bc}{bd} = \frac{a'c'}{a'd'} : \frac{b'c'}{b'd'}$$

$$\frac{ac}{ab} : \frac{dc}{db} = \frac{a'c'}{a'b'} : \frac{d'c'}{d'b'}$$

$$\frac{ab}{ad} : \frac{cb}{cd} = \frac{a'b'}{a'd'} : \frac{c'b'}{c'd'}$$

and any one of these implies the other two: and the same is true, obviously, for the reciprocals of these fractions.

(2) Also, if we denote the four radiants by A, B, C, D, and express by

* Might not these be the figures spoken of by Pappus (lib. vi., pr. 126), and upon which Dr. Simson offered some interesting conjectures in his letter to Dr. Jurin?—See *Dr. Trail's Life of Simson*, pp. 85-7.

AC the angle formed at S, the common origin of the radiants, by the lines A and C; and similarly with the other angles; then we shall have the following relations of the anharmonic ratios of the segments of the line ad and its corresponding angle AD:—

$$\frac{ac}{ad} : \frac{bc}{bd} = \frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD}$$

$$\frac{ac}{ab} : \frac{dc}{db} = \frac{\sin AC}{\sin AB} : \frac{\sin DC}{\sin DB}$$

$$\frac{ab}{ad} : \frac{cb}{cd} = \frac{\sin AB}{\sin AD} : \frac{\sin CB}{\sin CD}$$

This is proved at p. 241, Hutton, ii., to which the reader is referred.

(3) Again, if four planes which meet in one line be denoted by A, B, C, D, and be cut by any transversal in a, b, c, d: then denoting the dihedral angles formed by the planes A, C by AC, *etc.*, we shall have the equations in (2) also fulfilled by the segments of the transversal and the dihedral angle.

For, let the planes meet in the line ss', and through the point a draw a plane, P, perpendicular to ss', also through a draw a line in the plane P meeting the four planes in a, b', c', d' and ss' in s': and lastly, through $abcd$, $ab'c'd'$ draw another plane meeting ss' in s.

Then the angles formed by as', $b's'$, $c's'$ $d's'$ are the measures of the dihedral angles formed by the planes A, B, C, D, and we have

$$\frac{ac}{ad} : \frac{bc}{bd} = \frac{ac'}{ad'} : \frac{b'c'}{b'd'} = \frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD};$$

and similarly for the other two cases.

(4) Further, if A, B, C, D, four great circles meeting in one point on the sphere, be cut by any transversal great circle in a, b, c, d: then

$$\frac{\sin ac}{\sin ad} : \frac{\sin bc}{\sin bd} = \frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD}$$

$$\frac{\sin ac}{\sin ab} : \frac{\sin dc}{\sin db} = \frac{\sin AC}{\sin AB} : \frac{\sin DC}{\sin DB}$$

$$\frac{\sin ab}{\sin ad} : \frac{\sin cb}{\sin cd} = \frac{\sin AB}{\sin AD} : \frac{\sin CB}{\sin CD}$$

Hutton, ii., p. 76.

(5) If four straight lines A, B, C, D on a plane, or four great circles on a sphere, be cut by a transversal circle; viz. A in a, a', B in b, b', C in c, c', and D in d, d': then for the *arcs* ac, etc., we shall have

$$\frac{\sin\frac{1}{2}ac}{\sin\frac{1}{2}ad} : \frac{\sin\frac{1}{2}bc}{\sin\frac{1}{2}bd} = \frac{\sin\frac{1}{2}a'c'}{\sin\frac{1}{2}a'd'} ; \frac{\sin\frac{1}{2}b'c'}{\sin\frac{1}{2}b'd'}$$

$$\frac{\sin\frac{1}{2}ac}{\sin\frac{1}{2}ab} : \frac{\sin\frac{1}{2}dc}{\sin\frac{1}{2}db} = \frac{\sin\frac{1}{2}a'c'}{\sin\frac{1}{2}a'b'} : \frac{\sin\frac{1}{2}d'c'}{\sin\frac{1}{2}d'b'}$$

$$\frac{\sin\frac{1}{2}ab}{\sin\frac{1}{2}ad} : \frac{\sin\frac{1}{2}cb}{\sin\frac{1}{2}cd} = \frac{\sin\frac{1}{2}a'b'}{\sin\frac{1}{2}a'd'} : \frac{\sin\frac{1}{2}c'b'}{\sin\frac{1}{2}c'd'}$$

In plano. Let s be the intersection of the radiants: then joining the lines ac, bc, bd, da, $a'c'$, $b'c'$, $b'd'$, $d'a'$, we shall have four pairs of similar triangles sac, $sa'c'$, etc.: and hence

$$as : c's :: ac : a'c', \text{ from triangles } asc, a'sc'$$
$$ds : b's :: bd : b'd', \ldots\ldots\ldots\ldots\ bsd, b'sd'$$
$$d's : as :: a'd' : ad, \ldots\ldots\ldots\ldots\ asd, a'sd'$$
$$b's : cs :: b'c' : bc, \ldots\ldots\ldots\ldots\ bsd, b'sd';$$

whence, compounding, and recollecting that $ds \,.\, d's = c's \,.\, cs$, we shall have

$$ac \,.\, bd : ad \,.\, bc :: a'c' \,.\, b'd' : a'd' \,.\, b'c', \text{ or}$$

$$\frac{ac}{ad} : \frac{bc}{bd} = \frac{a'c}{a'd'} : \frac{b'c'}{b'd'}.$$

But $ac = 2 \sin \frac{1}{2}$ arc ac, etc.: which being substituted, we obtain the first of the formula enunciated; and the others are similarly deduced.

On the sphere. Conceive planes to be drawn through the centre of the sphere and the great circles in question, cutting the plane of the transversal circle. These sections will be the straight lines of the previous case, and their points of intersection with the transversal circle the same as the intersections of the great circles with the transversal circle. But by the preceding case the lines divide the circle in the manner expressed in the enunciation, and hence also the great circles divide it in the same manner.

The first case only is enunciated by Chasles, but he gives no demonstration of it.

(6) Lastly, let there be any three lines L, L′, L″, no two of which are situated in the same plane; then, it is known that innumerable straight lines can be drawn which shall touch all three (generating in fact, the hyperboloïd of one sheet): and it is a property of the segments $abcd$, $a'b'c'd'$, $a''b''c''d''$ of the lines L, L′ L″ made by any four lines which thus rest upon them, that their anharmonic ratios will be the same: viz.

$$\frac{ac}{ad} : \frac{bc}{bd} = \frac{a'c'}{a'd'} : \frac{b'c'}{b'd'} = \frac{a''c''}{a''d''} : \frac{b''c''}{b''d''}$$

$$\frac{ac}{ab} : \frac{dc}{db} = \frac{a'c'}{a'b'} : \frac{d'c'}{d'b'} = \frac{a''c''}{a''b''} : \frac{d''c''}{d''b''}$$

$$\frac{ab}{ad} : \frac{cb}{cd} = \frac{a'b'}{a'd'} : \frac{c'b'}{c'd'} = \frac{a''b''}{a''d''} : \frac{c''b''}{c''d''}$$

This also is established, nearly after Chasles's own manner, at p. 243, Hutton, vol. ii.

III.

From the preceding, the following properties are so easily deducible, as to merely need enumeration.

(1) When $\frac{ac}{ad} = \frac{bc}{bd}$ the division is the harmonical, and we have then

$$\frac{\sin AC}{\sin AD} = \frac{\sin BC}{\sin BD};$$

which is an elegant property of the harmonical system of radiants, but little

known to English readers. The same also holds on the sphere, for which see Leybourn's Repository, vol. vi., p. 70. On this subject, however, we shall say more hereafter.

(2) If there be four points, a, b, c, d, in a line, the anharmonic ratios of the angles made by lines drawn from them to all points whatever will be constantly the same.

(3) If through the same four points planes be drawn to meet in any straight lines whatever, the anharmonic ratios of the dihedral angles will be constantly the same.

(4) The same applies to the sines of the spherical angles formed by lines drawn from four points in a great circle to any point in the sphere.

(5) In any of the three preceding cases, the anharmonic ratios of the segments of any transersal whatever cutting any of the four radiants, will be constantly the same.

(6) If two lines be divided in the same anharmonic ratio, and three of the points of each, abc and $a'b'c'$, be upon three radiants, the fourth points d and d' will be upon a fourth radiant: and if abc, $a'b'c'$ be upon three planes which meet in one line, then d, d' will be in a fourth plane also passing through that line. And similarly on the sphere, for two great-circle transversals.

(7) When two lines are divided in the same anharmonic ratio, they may in innumerable ways be placed upon the same radiants, or upon the same radial planes.

(8) If any line $abcd$ be radially projected upon any plane or upon any line whatever the anharmonic ratio of the projection will be equal to that of the original line.*

IV.

Let a, b, c, d be the same points as before, in reference to the three cases, designated by *figs.* 1, 2, 3, before referred to, Hutton, p. 232): then we shall have

$$ab \,.\, cd + ac \,.\, bd - bc \,.\, ad = 0, \text{ in } fig. \ (1)$$
$$-ab \,.\, cd + ac \,.\, bd + bc \,.\, ad = 0, \ldots\ldots (2)$$
$$ab \,.\, cd - ac \,.\, bd + bc \,.\, ad = 0, \ldots\ldots (3)$$

(1) For, in *fig* 1, we have

$$\begin{aligned} ab \,.\, cd + ac \,.\, bd &= ac \,.\, bd + cd\,(bd + ca + cd) \\ &= ac \,.\, bd + bd \,.\, dc + cd \,.\, ca + cd^2 \\ &= (bd + dc)\,(ac + cd) \\ &= bc \,.\, ad. \quad \text{Whence, } etc. \end{aligned}$$

(2) In *fig*. 2, we have

$$\begin{aligned} ab \,.\, cd &= cd\,(ac + cb) \\ &= ac \,.\, cd + bc \,.\, cd \\ &= ac\,(db + bc) + bc\,(cb + bd) \\ &= ac \,.\, bd + bc\,(ac + bc + bd) \\ &= ac \,.\, bd + bc \,.\, ad. \quad \text{Whence, } etc. \end{aligned}$$

* This is the foundation of the projectibility of lines, the anharmonic ratio remaining the same; and it will be seen hereafter how important this simple principle becomes in the application of the method.

(3) In *fig*. 3, we have

$$\begin{aligned} ab\,.\,cd &= (ac - cb)(bd - bc) \\ &= ac\,.\,bd - bc\,(ac + bd - bc) \\ &= ac\,.\,bd - bc\,(ab + bd) \\ &= ac\,.\,bd - ad\,.\,bc. \quad \text{Whence, } etc. \end{aligned}$$

These three properties are in fact, like the former, but a single one, as will be seen by attending to the proper interchange of letters in passing from one figure to the next in succession. We may hence state it thus: (*fig*. 3,)

Any four points a, b, c, d, being taken *in order* in the same straight line, we shall have the relation

$$ab\,.\,cd + bc\,.\,ad = ac\,.\,bd.$$

This formula was, I think, first given by Euler in the New Petersburgh Commentaries, 1747-8: and two proofs (not essentially different from each other) were given of it.*

V.

By means of this theorem and the results of (II.) we may express the anharmonic ratio in terms of the segments of *two* lines cutting the same radiants, which will frequently be useful:—

$$\frac{ac}{ad} : \frac{bc}{bd} - \frac{a'b'}{a'd'} : \frac{c'b'}{c'd'} = 1 \qquad \frac{a'c'}{a'd'} : \frac{b'c'}{b'd'} - \frac{ab}{ad} : \frac{cb}{cd} = 1$$

$$-\frac{ad}{ab} : \frac{cd}{cb} + \frac{a'c'}{a'b'} : \frac{d'c'}{d'b'} = 1 \qquad -\frac{a'd'}{a'b'} : \frac{c'd'}{c'b'} + \frac{ac}{ab} : \frac{dc}{db} = 1$$

$$\frac{ab}{ac} : \frac{db}{dc} + \frac{a'd'}{a'c'} : \frac{b'd'}{b'c'} = 1 \qquad \frac{a'b'}{a'c'} : \frac{d'b'}{d'c'} + \frac{ad}{ac} : \frac{bd}{bc} = 1;$$

and any one of these equations, like any one of the three former, implies the other two.

For by the theorem of Euler, in the last paragraph, we have

$$\begin{aligned} 1 = \frac{ac.bd}{bc.ad} - \frac{ab.cd}{bc.ad} &= \frac{ac}{ad} : \frac{bc}{bd} - \frac{ab}{ad} : \frac{cb}{cd} \\ &= \frac{ac}{ad} : \frac{bc}{bd} - \frac{a'b'}{a'd'} : \frac{c'b'}{c'd'}; \end{aligned}$$

and in the same way, all the rest are deducible from the same fundamental equation, combined with the general property of the anharmonic system of radiants.†

* This paper of Euler's is curious, as shewing the very limited state of *geometrical* knowledge which existed amongst the Continental mathematicians of that period: for, except this very simple property, there is perhaps not a single theorem contained in that paper, which was not familar to an extensive class of our countrymen at that time.

† Chasles has given these equations differently as to signs, but it would appear that the results here obtained are the correct forms. Besides, of the three equations given at page 304 *Aperçu Historique*, the third is, in reality, only a repetition of the first; instead of being, as the subject required, an *independent form* of expression.

VI.

Besides the foregoing, M. Chasles has given another form for the anharmonic ratio, which he thus enunciates, and to which I annex an investigation by the co-ordinate method, he not having offered one of any kind:—

Let there be two given lines in the same plane; and let there be given any two points O, E, upon the former, and any two O′, E′, upon the latter: and about two given poles P, P′ let there revolve two straight lines, cutting the two former in a, a' respectively, so that the equation (λ, μ being constants)

$$\lambda . \frac{\mathrm{O}a}{\mathrm{E}a} + \mu . \frac{\mathrm{O}'a'}{\mathrm{E}'a'} = 1$$

shall be fulfilled: then the two revolving lines intersecting in Q, the point Q will describe a conic section, which passes through the poles P and P′.

Let the given lines OE, O′E′ meet in S, and take them as co-ordinate axes: and put

$$\mathrm{SO} = a, \quad \mathrm{SO}' = a', \quad \mathrm{SE} = b, \quad \mathrm{SE}' = b' ;$$

and denote the points P, P′ by $\alpha\beta$, $\alpha'\beta'$.

Then the lines Pa and P′a' may be expressed by

$$y - \beta = p\,(x - \alpha) \dots\dots\dots\dots (1)$$
$$y - \beta' = p'(x - \alpha') \dots\dots\dots\dots (2)$$

When $y = 0$ in (1), and $x = 0$ in (2) we get

$$x = \mathrm{S}a = \frac{\beta - p\alpha}{p} \dots\dots\dots\dots (3)$$
$$y = \mathrm{S}a' = \beta' - p'\alpha' \dots\dots\dots\dots (4)$$

Hence the equation of condition becomes

$$\lambda . \frac{a - \dfrac{\beta - p\alpha}{p}}{b - \dfrac{\beta - p\alpha}{p}} + \mu . \frac{a' - (\beta' - p'\alpha')}{b' - (\beta' - p'\alpha')} = 1, \text{ or}$$

$$\lambda . \frac{(a + \alpha)p - \beta}{(b + \alpha)p - \beta} + \mu . \frac{(a' + p'\alpha') - \beta'}{(b' + p'\alpha') - \beta'} = 1 \dots\dots\dots \quad (5)$$

Now if from (1, 2) we insert the values of p and p' in (5), we shall obtain after simple reduction,

$$\lambda . \frac{(a + \alpha)(y - \beta) - \beta\,(x - \alpha)}{(b + \alpha)(y - \beta) - \beta\,(x - \alpha)} + \mu . \frac{(a' - \beta')(x - \alpha') + \alpha'(y - \beta')}{(b' - \beta')(x - \alpha') + \alpha'(y - \beta')} = 1 \,..(6)$$

which is, being of the second degree, the equation of a conic section.

If we multiply out the denominator every term of the result will involve one of the factors $x - \alpha$ or $y - \beta'$: and hence the condition will be fufilled $x - \alpha = 0$ and $y - \beta = 0$ simultaneously; that is, $\alpha\beta$ is a point in the curve. In like manner each term will involve one of the factors $x - \alpha'$, or $y - \beta'$, and hence, also, $\alpha'\beta'$ is a point in the curve.

The connection of this proposition with the anharmonic ratio will be shewn in a future part of this series of papers; and a demonstration more in accordance with the method pursued in these researches will also be given.

CHAPTER II.

THE SECTION OF INVOLUTION.

VII.

Definition 1. Let there be taken, estimated from a given point o, in a right line, any number of points a, b, c, *etc.*, and as many others a', b', c', *etc.*, so related to the former ones that

$$ao \, . \, oa' = bo \, . \, ob' = co \, . \, oc' = etc. :$$

then these points are said to be *in involution*, and the line itself, so divided, is said to constitute a *section of involution.*

Def. 2. The point o is called the *centre of involution.*

Def. 3. If two points ω, ω' be taken equally distant from o on each side of it, so that $\omega o \, . \, o\omega' = ao \, . \, oa'$, then ω, ω', are called the *foci of involution.*

The definition here employed of the *section of involution*, was first noticed by Chasles as a *property* of the division so named by Desargues, or of the property proved by Pappus (*Math. Coll.* vii. p. 130). Chasles himself deduces the properties of Pappus and Desargues from his own anharmonic ratio, with much elegance: but the very simple mode of constructing the division of involution which this definition affords, by means hereafter to be explained, appears to render that chosen here the most appropriate for our purpose. It will also enable us to remove a certain degree of obscurity and unsatisfactoriness, which I could not but feel, myself, in my first study of Chasles's own discussion of the subject.

VIII.

1. Let there be three pairs of conjugates, a and a', b and b', c and c', in involution, and let m be any other point in the same line; also, let α, β, γ be the respective middles of aa', bb', cc', and let them lie in the same order: then we shall have

$$ma \, . \, ma' \, . \, \beta\gamma - mb \, . \, mb' \, . \, \gamma\alpha + mc \, . \, mc' \, . \, \alpha\beta = 0.$$

$a \quad b \quad c \quad a' \quad b' \quad c' \quad m$

$\alpha \quad \beta \quad o \quad \gamma$

First, let m be at the centre of involution: then the property becomes

$$ao \, . \, oa' \, . \, \beta\gamma - bo \, . \, ob' \, . \, \gamma\alpha + co \, . \, oc' \, . \, \alpha\beta = 0,$$

and in consequence of the definition,

$$ao \, . \, oa' = bo \, . \, ob' = co \, . \, oc' :$$

and by the structure of the line,

$$\beta\gamma - \gamma\alpha + \alpha\beta = 0,$$

and the conclusion is established.

Second, let m be any where in the line, as suppose to the right of c': then

$$ma = mo + oa, \text{ and } ma' = mo - oa', \text{ or}$$

$$ma \, . \, ma' = mo^2 + mo\,(oa - oa') + oa \, . \, oa'$$

$$= mo^2 + 2mo \, . \, o\alpha + oa \, . \, oa'; \text{ and similarly,}$$

$$mb \, . \, mb' = mo^2 + 2mo \, . \, o\beta + ob \, . \, ob', \text{ and}$$

$$mc \, . \, mc' = mo^2 + 2mo \, . \, o\gamma + oc \, . \, oc'.$$

Wherefore the enunciated expression becomes

$$0 = \{oa \,.\, oa' \,.\, \beta\gamma - ob \,.\, ob' \,.\, \gamma\alpha + oc \,.\, oc' \,.\, \alpha\beta\}$$
$$+ 2mo\{o\alpha \,.\, \beta\gamma - o\beta \,.\, \gamma\alpha + o\gamma \,.\, \alpha\beta\}$$
$$+ mo^2\{\beta\gamma - \gamma\alpha + \alpha\beta\},$$

Now of three bracketed factors of this expression, the first has been shewn to be zero in the former case: the second is zero by Euler's theorem (p. 175): and the third is zero by identity. Whence the entire value is zero, as enunciated.

2. This conclusion may also be expressed without the aid of α, β, γ: for since

$\beta\gamma = \frac{1}{2}(bc + b'c')$, $\gamma\alpha = \frac{1}{2}(ca + c'a')$, and $\alpha\beta = \frac{1}{2}(ab + a'b')$ we have, by substitution,

$$ma \,.\, ma'(bc+b'c') - mb \,.\, mb'(ca+c'a') + mc \,.\, mc'(ab+a'b') = 0.$$

3. If the same hypothesis be made, we shall also have this relation between $a, b, c, \alpha, \beta, \gamma$:—

$$a\alpha^2 \,.\, \beta\gamma - \beta b^2 \,.\, \gamma\alpha + \gamma c^2 \,.\, \alpha\beta = \alpha\beta \,.\, \beta\gamma \,.\, \gamma\alpha.$$

For, we have

$$m\alpha^2 \,.\, \beta\gamma - m\beta^2 \,.\, \gamma\alpha + m\gamma^2 \,.\, \alpha\beta = \alpha\beta \,.\, \beta\gamma \,.\, \gamma\alpha :*$$

from which subtract the previous equation, and we get

$$(m\alpha^2 - ma \,.\, ma')\beta\gamma - (m\beta^2 - mb \,.\, mb')\gamma\alpha + (m\gamma^2 - mc . mc')\alpha\beta = \alpha\beta . \beta\gamma . \gamma\alpha.$$

But $ma \,.\, ma' = (m\alpha + \alpha a)(m\alpha - \alpha a) = m\alpha^2 - \alpha a^2$; whence

$$m\alpha^2 - ma \,.\, ma' = \alpha a^2;$$

Similarly, $m\beta^2 - mb \,.\, mb' = \beta b^2$;

And, $m\gamma^2 - mc \,.\, mc' = \gamma c^2$;

Wherefore, substituting these, we get the enunciated equality.

IX.

If three points a, b, c, and their conjugates a', b', c', form an involution of six points, the following seven relations will subsist among the segments:—

$$ab \,.\, ab' : ac \,.\, ac' = a'b' \,.\, a'b : a'c' \,.\, a'c$$
$$bc \,.\, bc' : ba \,.\, ba' = b'c' \,.\, b'c : b'a' \,.\, b'a$$
$$ca \,.\, ca' : cb \,.\, cb' = c'a' \,.\, c'a : c'b' \,.\, c'b$$
$$ab \,.\, b'c' \,.\, ca' = a'b' \,.\, bc \,.\, c'a$$
$$bc \,.\, c'a' \,.\, ab' = b'c' \,.\, ca \,.\, a'b$$
$$ca \,.\, a'b' \,.\, bc' = c'a' \,.\, ac \,.\, b'c$$
$$ab \,.\, bc' \,.\, ca' = a'b \,.\, b'c \,.\, c'a$$

* This property is a particular case of the second of *Stewart's General Theorem*, (viz. when the vertex of the triangle in the enunciation coalesces with the base,) and as such it is quoted by Chasles. It, however, admits of a very simple proof as a property of four points in a line, as follows:

$$\alpha m^2 - \beta m^2 = (\alpha m - \beta m)(\alpha m + \beta m) = \alpha\beta \,.\, \alpha m + \alpha\beta \,.\, \beta m,$$
$$\beta m^2 - \gamma m^2 = (\beta m - \gamma m)(\beta m + \gamma m) = \beta\gamma \,.\, \beta m + \beta\gamma \,.\, \gamma m.$$

Multiply the former by $\beta\gamma$, and the latter by $\alpha\beta$, and subtract: then there results the expression in question.—This method was given by Mr. Rutherford in the Lady's and Gentleman's Diary, 1842, p. 48-9.

The first three are deduced in precisely the same manner, and it will suffice to give the investigation of one, which may be the first of them.

From the definition we get

$$oa\,.\,oa'=ob\,.\,ob',\ \text{or}\ ob:oa::oa':ob'.$$

$$\begin{array}{ll|ll} \therefore\ oa-ob:ob'-oa'::ob:oa & & \text{and}\ ob+oa':ob'+oa::ob:oa & \\ \text{or}\qquad ab:\ \ a'b'\ ::ob:oa & & \text{or}\qquad a'b\ :\ \ ab'\ ::ob:oa. & \end{array}$$

Whence, compounding these two, we get

$$ab\,.\,ab':a'b'\,.\,a'b::oa':oa.$$

In a similar manner from $oa\,.\,oa'=oc\,.\,oc'$, we have

$$ac\,.\,ac':a'c'\,.\,a'c::oo':oa.$$

Whence, $\qquad ab\,.\,ab':a'b'\,.\,a'b::ac:ac':a'c'\,.\,a'c.$

Which is the first of the equations in first system, at once. The other two are found by treating the two pairs of equations of the definition in a similar manner.

$$ob\,.\,ob'=oc\,.\,oc',\ ob\,.\,ob'=oa\,.\,oa',\ \text{and}$$
$$oc\,.\,oc'=oa\,.\,oa',\ oc\,.\,oc'=ob\,.\,ob':$$

that system being compared with each of the other two, which is to have one of its points of section in every term of the expression.

From the first three equations we may deduce the last four by mere composition. To secure order in the processes, denote the first and second sides of the first equation by a_1 and a_2; those of the second equation by b_1 and b_2; and those of the third equation by c_1 and c_2: then all the combinations in *threes* which are essentially different in their composition, are the four following,—

$$a_1b_1c_2=a_2b_2c_1,$$
$$a_1b_2c_1=a_2b_1c_2,$$
$$a_1b_2c_2=a_2b_2c_1,$$
$$a_1b_1c_1=a_2b_2c_2.$$

The compositions being made according to this model, we shall obtain the four remaining properties of the section of involution stated in the enunciation of the theorem. It will be sufficient here to give one single case, the others being exactly similar, and formed according to the model above.

$$\mathrm{ab}\,.\,ab'\,.\,a'c'\,.\,\mathrm{a'c}=\mathrm{a'b'}\,.\,a'b\,.\,ac\,.\,\mathrm{ac'}$$
$$bc\,.\,\mathrm{bc'}\,.\,\mathrm{b'a'}\,.\,b'a=b'c'\,.\,\mathrm{b'c}\,.\,\mathrm{ba}\,.\,ba'$$
$$c'a'\,.\,\mathrm{c'a}\,.\,cb\,.\,\mathrm{cb'}=\mathrm{ca'}\,.\,ca\,.\,c'b'\,.\,\mathrm{c'b}$$

in which the factors in roman cancel by composition, and each of the others is repeated twice, giving

$$bc^2.\,c'a'^2.\,ab'^2=b'c'^2.\,ca^2.a'b^2,$$

which, on extracting the root, is identical with the second equation of the second series.

Cor. 1. It will obviously follow, that if there be any number of pairs of points which are in involution, then *any three pairs* will possess the properties above deduced.

This is a property of considerable importance in the applications of the theory.

Cor. 2. Any one of these seven equations implies the other six: for each

one implies (*see* Hutton, ii. pp. 230-1,) that the line is divided in involution; and this again that all the properties have place simultaneously, by the demonstration above.

Scholium 1. The first three of these equations were discovered by Desargues, an architect of Lyons; but this great man, though honoured by the friendship of Descartes and tutorage of Pascal, and worthily designated by Poncelet as the "Monge of his age," failed to attract the attention of his contemporaries or immediate successors to his remarkable investigations. Not a single copy of a single work of his on these subjects is known now to exist: and it is only from the sarcastic account of them by Beaugrand, that their existence and character is known to modern geometers! Pascal's MS. on the Conic Sections, too, is lost: but all that remains to tell us the character of that loss, may be seen in Leybourn's Mathematical Repository, N. S., vol. vi. pp. 3-8.

The four last properties were known to Pappus, who gives them amongst the Lemmas for the treatises of Apollonius. They are virtually expressed in the Mathematical Collections, book vii. prop. 130.

The entire series, however, were known to these geometers as properties of the segments of a transversal cutting certain figures, to which we shall hereafter refer with some detail—and not as properties of a line divided according to our definition of the *section of involution.*

It may be desirable here to state, that the notation employed in Hutton ii. has here been changed. The notation of this paper is the same as that of Chasles, except (for a reason which will presently be explained) that the small letters a, b, c, etc. are used instead of the capitals A, B, C, etc. The following scale will enable us to convert the one system of notation to the other by inspection: the letters in the upper line in Hutton corresponding to the lower one in Chasles, as here changed:—

A C E O B D F (Hutton)
$a\ b\ c\ o\ a'\ b'\ c'$ (Chasles).

Scholium 2. The construction of the formulæ of Desargues is too obvious to need any memorial rule for writing them: and so, also in the arrangement here given, are those of Apollonius. Chasles, however, has given the following rule:—

"When we take three points a, b, c, which belong to three couples, each of them makes two segments with the conjugates of the two others, thus giving six segments: and the product of three of these segments, which have no common extremity, is equal to the product of the three others."

X.

If three points a, b, c, and their conjugates a', b', c', be in involution, the the following anharmonic ratios will be equal; and conversely, if one of these anharmonic ratios be equal to its conjugate one, the six points are in involution: viz. the anharmonic ratios of the systems

(1) $a'bca$ are equal to $ab'c'a'$;
(2) $b'cab$ .... $bc'a'b'$;
(3) $c'abc$ .... $ca'b'c'$;
(4) $aa'b'c$.......... $a'abc'$;
(5) $bb'c'a$.......... $b'bca'$;
(6) $cc'a'b$ .... $c'cab'$.

1. For from the equations of Desargues, in the last article, we have,

$$\frac{ab}{ac} : \frac{a'b}{a'c} = \frac{a'b'}{a'c'} : \frac{ab'}{ac'},$$
$$\frac{bc}{ba} : \frac{b'c}{b'a} = \frac{b'c'}{b'a'} : \frac{bc'}{ba'},$$
$$\frac{ca}{cb} : \frac{c'a}{c'b} = \frac{c'a'}{c'b'} : \frac{ca'}{cb'};$$

which indicate the truth of the first three parts of the proposition.

2. Again, divide the first of the Apollonian equations by aa', the second by bb', and the third by cc': then we get after forming the proportions,

$$\frac{ab}{aa'} : \frac{cb}{ca'} = \frac{a'b'}{a'a} : \frac{c'b'}{c'a},$$
$$\frac{bc}{bb'} : \frac{ac}{ab'} = \frac{b'c'}{b'b} : \frac{a'c'}{a'b},$$
$$\frac{ca}{cc'} : \frac{ba}{bc'} = \frac{c'a'}{c'c} : \frac{b'a'}{b'c};$$

which shew that the three last sets are in the same anharmonic ratios, each with each.

The converse follows from this consideration: that if any one of these relations have place, the line is divided in involution; and that a line divided in involution has *all* the properties which have been here proved to belong to its segments.

Corollary. Since each equality of anharmonic ratios implies two others, there will arise eighteen equations in all: and as it will be convenient in the future use of these properties to have them tabulated, the reader is recommended to form such a table for himself.

Scholium. The symmetrical ordering of the letters expressing the different points will contribute to the easy remembrance of these relations, and operate as a check upon any oversight in writing down the formulæ implied in them. It is mainly for this purpose, in fact, that I have departed from the precise forms of M. Chasles: but still these forms would not have been so easily deduced but for the entire change which has here been made from the order and method of investigation employed by that distinguished geometer.

(To be continued.)

ON A CASE OF ELIMINATION WITH AN EXAMPLE.

[*Mr. James Anderson, Montrose.*]

Let $P = 0$, and $Q = 0$, be two equations, in which P and Q are functions of x and y, and of their differential coefficients with respect to t; it is here proposed to point out the method of eliminating t, and thereby finding the relation subsisting among x and y, and the differential coefficients of y with respect to x.

We may assume $P = 0$ and $Q = 0$ to be differential equations of the same order; for if they were of different orders, it would be easy to reduce the

lower by differentiation to the same order as the higher. This being premised, it is necessary to substitute instead of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, etc., their values in terms of $\frac{dt}{dx}$, $\frac{d^2t}{dx^2}$, etc., conjoined in the case of $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, *etc.*, with the differential coefficients of y with respect to x; by this process we shall obtain two equations

$$P_1 = 0; \quad Q_1 = 0;$$

in which P_1 and Q_1 are functions of x, y, and the differential coefficients of t and y with respect to x. It is also manifest that these equations will be of the same order as the original equations, and that in each the highest differential coefficient of t with respect to x, suppose $\frac{d^n t}{dx^n}$, will be the same. Eliminating $\frac{d^n t}{dx^n}$ between these two equations, there will result an equation, in which there cannot be a higher differential coefficient of t than $\frac{d^{n-1} t}{dx^{n-1}}$; this equation we shall designate by

$$P_2 = 0.$$

Differentiating this with respect to x, until we obtain a term $\frac{d^n t}{dx^n}$, we shall have another equation, by means of which and either of the equations $P_1 = 0$, or $Q_1 = 0$, we may eliminate $\frac{d^n t}{dx^n}$, so as to have an equation

$$Q_2 = 0;$$

in which there is no coefficient higher than $\frac{d^{n-1} t}{dx^{n-1}}$. There are now two equations $P_2=0$, $Q_2=0$, either of the same order or easily reduced by differentiation to the same order, and at least one degree lower than $P_1=0$, $Q_1=0$. Treating these in the same manner, other two equations of a still lower order in respect to t may be found. The process being continued, we shall at last arrive at two equations in which no higher differential coefficients of t than $\frac{dt}{dx}$ will be found, and by the elimination of these we obtain finally an equation in which t is not found. It may also be observed that if $\frac{d^n y}{dt^n}$ appear in the original equations, there will in general be ultimately a term $\frac{d^{2n-1}y}{dx^{2n-1}}$.

As an example to this simple theory, we may consider the well known differential equations of motion

$$\frac{d^2x}{dt^2} = X; \quad \frac{d^2y}{dt^2} = Y.$$

Now $$\frac{d^2x}{dt^2} = \frac{d}{dt}\cdot\frac{dx}{dt} = \frac{d}{dx}\left(\frac{1}{\frac{dt}{dx}}\right)\frac{dx}{dt} = -\frac{\frac{d^2t}{dx^2}}{\frac{dt^3}{dx^3}};$$

and $$\frac{d^2 y}{dt^2} = \frac{d}{dt} \cdot \frac{dy}{dt} = \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\frac{dt}{dx}}\right)\frac{dx}{dt} = \frac{\frac{dt}{dx} \cdot \frac{d^2y}{dx^2} - \frac{dy}{dx} \cdot \frac{d^2t}{dx^2}}{\frac{dt^3}{dx^3}}.$$

Hence $$-\frac{\frac{d^2t}{dx^2}}{\frac{dt^3}{dx^3}} = X; \text{ and } \frac{\frac{dt}{dx} \cdot \frac{d^2y}{dx^2} - \frac{dy}{dx} \cdot \frac{d^2t}{dx^2}}{\frac{dt^3}{dx^3}} = Y;$$

from which by the elimination of $\frac{d^2t}{dx^2}$, there results

$$-X\frac{dt^3}{dx^3} \cdot \frac{dy}{dx} = \frac{dt}{dx} \cdot \frac{d^2y}{dx^2} - Y\frac{dt^3}{dx^3};$$

or $$\frac{dt^2}{dx^2} = \frac{\frac{d^2y}{dx^2}}{Y - X\frac{dy}{dx}} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Before proceeding further in the elimination of t, it is worth while to attend to the result here obtained. By inverting,

$$\frac{dx^2}{dt^2} = \frac{Y - X\frac{dy}{dx}}{\frac{d^2y}{dx^2}}:$$

And in like manner we may find

$$\frac{dy^2}{dt^2} = \frac{X - Y\frac{dx}{dy}}{\frac{d^2x}{dy^2}} = \frac{dy^2}{dx^2} \cdot \frac{Y - X\frac{dy}{dx}}{\frac{d^2y}{dx^2}}.$$

Adding these two equations, and considering that $\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = \frac{ds^2}{dt^2}$, we get

$$\frac{ds^2}{dt^2} = \frac{\left(1 + \frac{dy^2}{dx^2}\right)\left(Y - X\frac{dy}{dx}\right)}{\frac{d^2y}{dx^2}} = \frac{ds^3}{dx^3} \cdot \frac{Y\frac{dx}{ds} - X\frac{dy}{ds}}{\frac{d^2y}{dx^2}}.$$

Now $\dfrac{\frac{ds^3}{dx^3}}{\frac{d^2y}{dx^2}} = \rho$, the radius of curvature, and $Y\frac{dx}{ds} - X\frac{dy}{ds}$ is the normal component of the forces acting upon the body in motion, which may be denoted by N: hence

$$(\text{velocity})^2 = V^2 = \rho N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

an equation much used before the introduction of the rectangular resolution

of force by Maclaurin, and which also shows the measure of centrifugal force, since the centrifugal force must be equal and opposite to the normal component of the forces producing motion.

Returning to (1), and differentiating,

$$2\frac{dt}{dx}\cdot\frac{d^2t}{dx^2}=\frac{d}{dx}\left(\frac{\frac{d^2y}{dx^2}}{Y-X\frac{dy}{dx}}\right)=-2X\frac{dt^4}{dx^4}=-2X\left(\frac{\frac{d^2y}{dx^2}}{Y-X\frac{dy}{dx}}\right)^2.$$

This equation is the same as

$$\frac{d}{dx}\left(\frac{Y-X\frac{dy}{dx}}{\frac{d^2y}{dx^2}}\right)=2X\ldots\ldots\ldots\ldots\ldots(3)$$

so that the elimination of t is completely effected.

We might find in the same manner

$$\frac{d}{dy}\left(\frac{X-Y\frac{dx}{dy}}{\frac{d^2x}{dy^2}}\right)=2Y\ldots\ldots\ldots\ldots\ldots(4)$$

Integrating (3) and (4) with respect to x and y respectively, and adding the results

$$\frac{Y-X\frac{dy}{dx}}{\frac{d^2y}{dx^2}}+\frac{X-Y\frac{dx}{dy}}{\frac{d^2x}{dy^2}}=2\int(Xdx+Ydy)+C;$$

which is readily transformed into

$$\frac{ds^3}{dx^3}\cdot\frac{Y\frac{dx}{ds}-X\frac{dy}{ds}}{\frac{d^2y}{dx^2}}=2\int(Xdx+Y\,dy)+C.$$

Hence by (2) and the last result,

$$V^2=\rho N=2\int(Xdx+Ydy)+C\ldots\ldots\ldots\ldots\ldots(5)$$

a well known form, and which might have been obtained at once from the original equations.

The formula marked (3) may be used in the solution of the following problem: A material particle, moving in a resisting medium, is acted upon by two rectangular forces X′ and Y′; required the law of resistance of the medium that the particle may move in any given curve.

The law of the resistance must be some function of the velocity, and may therefore be represented by $\phi\left(\frac{ds}{dt}\right)$. Hence we shall have

$$X=X'-\phi\left(\frac{ds}{dt}\right)\cdot\frac{dx}{ds};\text{ and }Y=Y'-\phi\left(\frac{ds}{dt}\right)\cdot\frac{dy}{ds};$$

from these we find that $Y - X\frac{dy}{dx}$ is the same as $Y' - X'\frac{dy}{dx}$; hence by (3)

$$\frac{d}{dx}\left\{\frac{Y' - X'\frac{dy}{dx}}{\frac{d^2y}{dx^2}}\right\} = 2X' - 2\frac{dx}{ds}\cdot\phi\left(\frac{ds}{dt}\right):$$

$$\text{and } \phi\left(\frac{ds}{dt}\right) = \tfrac{1}{2}\left\{2X' - \frac{d}{dx}\left(\frac{Y' - X'\frac{dy}{dx}}{\frac{d^2y}{dx^2}}\right)\right\}\cdot\frac{ds}{dx}:$$

so that the law of resistance is determined.

In the second chapter of the first book of the Mecanique Celeste, a particular case of this problem is considered: viz, that in which the force is gravity. Making the axis of x horizontal, we have in this case

$$X' = 0; \text{ and } Y' = g;$$

the axis of y being supposed directed vertically downwards. Substituting these values, there results

$$\phi\left(\frac{ds}{dt}\right) = \tfrac{1}{2}\left\{-g\frac{d}{dx}\left(\frac{1}{\frac{d^2y}{dx^2}}\right)\right\}\cdot\frac{ds}{dx} = \frac{g\frac{d^3y}{dx^3}\cdot\frac{ds}{dx}}{2\left(\frac{d^2y}{dx^2}\right)^2},$$

the same result as that given in the Mecanique Celeste, allowance being made for a difference of notation.

The same formula (3) may also be applied to the solution of this problem: Given the curve which a particle describes and one of the rectangular components of the force or forces, acting upon it, to find what must be the value of the other rectangular component.

In this example X will either be a given function of x alone, or by means of the equation of the curve may be changed into a function of x alone. Integrating (3) with respect to x,

$$\frac{Y - X\frac{dy}{dx}}{\frac{d^2y}{dx^2}} = 2\int X dx + C.$$

By what we have already seen, $\dfrac{Y - X\frac{dy}{dx}}{\frac{d^2y}{dx^2}} = v^2.\dfrac{dx^2}{ds^2}$;

hence supposing V, a, and a, to be simultaneous values of the velocity, the inclination of the trajectory to the axis of x, and of the axis of x itself, we shall have

$$V^2\cos^2 a = 2\int_a X dx + C.$$

$$\text{Whence}\ \frac{Y - X\dfrac{dy}{dx}}{\dfrac{d^2y}{dx^2}} = V^2\cos^2\alpha + 2\int_a^x X\,dx \quad\text{.....................(6)}$$

$$\text{And}\ Y = X\frac{dy}{dx} + \frac{d^2y}{dx^2}\left\{V^2\cos^2\alpha + 2\int_a^x X\,dx\right\}.$$

As an example of the use of this formula, we may take a problem from Walton's Mechanical Problems, p. 180, which may be thus shortly enunciated:—

A particle is projected with a given velocity at right angles from a plane, to which it is attracted by a force varying as the distance from it; what is the force parallel to the plane which will compel the particle to move in a parabola, whose axis is in the plane.

Making the axis of the parabola the axis of y, we have $X = -\mu x$: also 0, 0, and V are simultaneous values of a, x, and v; hence

$$Y = -\mu x\frac{dy}{dx} + \frac{d^2y}{dx^2}\left\{V^2 + 2\int_0^x -\mu x\,dx\right\}.$$

The equation of the parabola is $x^2 = py$, from which deducing $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, we obtain,

$$Y = -\frac{2\mu x^2}{p} + \frac{2}{p}\left(V^2 - \mu x^2\right) = \frac{2V^2}{p} - \frac{4\mu x^2}{p} = \frac{2V^2}{p} - 4\mu y,$$

the result there given.

If we applied (3) to the parabola, whose equation is $x^2 = py$, we should have

$$\frac{p}{2}\cdot\frac{d}{dx}\left(Y - \frac{2xX}{p}\right) = 2X,$$

$$\text{or, } p\,.\frac{dY}{dx} = 6X + 2x\frac{dX}{dx} = \frac{2}{x^2}\cdot\frac{d}{dx}(x^3X).$$

Hence $px^2.\ \frac{dY}{dx} = 2\ \frac{d}{dx}(x^3X)$, is a characteristic of all forces producing a parabolic orbit. Thus if Y were constant, we should have

$$x^3X = c, \text{ and } X = \frac{c}{x^3},$$

where c would necessarily be either zero or negative.

It is not necessary here to consider the various formulæ connected with central forces, as they are well known. There is one case, however, which may be resolved by (6) without transformation into polar co-ordinates. Let the central force be attractive and proportional to the distance of the projectile; then we have

$$X = -\mu x, \text{ and } Y = -\mu y: \text{ hence,}$$

$$\frac{-y + x\dfrac{dy}{dx}}{\dfrac{d^2y}{dx^2}} = \frac{V^2}{\mu}\cos^2\alpha - 2\int_a^x x\,dx = \frac{V^2}{\mu}\cos^2\alpha + a^2 - x^2.$$

$$\text{But } \frac{d^2y}{dx^2} = \frac{1}{x}\cdot\frac{d}{dx}\left(-y + x\frac{dy}{dx}\right):$$

$$\therefore \frac{\dfrac{d}{dx}\left(-y+x\dfrac{dy}{dx}\right)}{-y+x\dfrac{dy}{dx}} = \frac{x}{\dfrac{V^2}{\mu}\cos^2 a + a^2 - x^2}.$$

Integrating

$$\log\left(-y+x\frac{dy}{dx}\right) = -\tfrac{1}{2}\log\left(\frac{V^2}{\mu}\cos^2 a + a^2 - x\right) + C,$$

and $\log(-b + a\tan a) = -\tfrac{1}{2}\log\left(\dfrac{V^2}{\mu}\cos^2 a\right) + C$:

whence,

$$-y + x\frac{dy}{dx} = \frac{V\cos a\,(-b + a\tan a)\mu^{-\frac{1}{2}}}{\left(\dfrac{V^2}{\mu}\cos^2 a + a^2 - x^2\right)^{\frac{1}{2}}} = \frac{A}{(c^2-x^2)^{\frac{1}{2}}}.$$

Now, $-y + x\dfrac{dy}{dx} = x^2\dfrac{d}{dx}\left(\dfrac{y}{x}\right)$; hence $\dfrac{d}{dx}\left(\dfrac{y}{x}\right) = \dfrac{A}{x^2\sqrt{c^2-x^2}}$, $\therefore$

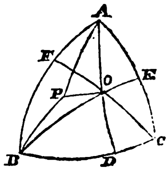
$$\frac{y}{x} = \int\frac{A\,dx}{x^2\sqrt{c^2-x^2}} = \int\frac{A\,dx}{x^3\sqrt{c^2x^{-2}-1}} = -\frac{A}{2}\int\frac{d\cdot\frac{1}{x^2}}{\sqrt{c^2x^{-2}-1}} = -A'\sqrt{c^2x^{-2}-1};$$

whence $y = -A'\sqrt{c^2-x^2}$, the equation of an ellipse, the centre being at the origin.

PROPERTIES OF THE SPHERICAL TRIANGLE.

[*Mr. Philip Beecroft.*]

Proposition A.



Let O be any given point within a spherical triangle ABC, and describe great circles to pass through it and each of the angles of the triangle; then, if the angles BOC, COA, AOB be denoted by a_1, a_2, a_3 respectively, and the angles POA, POB, POC which any great circle passing through O makes with the lines OA, OB, OC, by θ_1, θ_2, θ_3 respectively, we shall have

$$\sin a_1\cos\theta_1 + \sin a_2\cos\theta_2 + \sin a_3\cos\theta_3 = 0.$$

For, $a_1 = \theta_3 - \theta_2$, $a_2 = 2\pi - \theta_1 - \theta_3$, $a_3 = \theta_1 + \theta_2$;

$$2\sin a_1\cos\theta_1 = \sin(a_1+\theta_1) + \sin(a_1-\theta_1) = \sin(\theta_1+\theta_3-\theta_2) - \sin(\theta_1+\theta_2-\theta_3)$$

$$\begin{aligned}2\sin a_2\cos\theta_2 &= \sin(a_2+\theta_2) + \sin(a_2-\theta_2)\\ &= \sin(2\pi-\theta_1-\theta_3+\theta_2) + \sin(2\pi-\theta_1-\theta_2-\theta_3)\\ &= -\sin(\theta_1+\theta_3-\theta_2) - \sin(\theta_1+\theta_2+\theta_3)\end{aligned}$$

$$2\sin a_3\cos\theta_3 = \sin(a_3+\theta_3) + \sin(a_3-\theta_3) = \sin(\theta_1+\theta_2+\theta_3) + \sin(\theta_1+\theta_2-\theta_3).$$

Hence, by addition,

$$\sin a_1\cos\theta_1 + \sin a_2\cos\theta_2 + \sin a_3\cos\theta_3 = 0 \ldots\ldots\ldots\ldots (a)$$

Let three other great circles respectively pass through one of the angles A, B, C, and mutually intersect in the point P; then putting

$$OA = d_1,\ OB = d_2,\ OC = d_3,\ PA = \delta_1,\ PB = \delta_2,\ PC = \delta_3,$$

and substituting the values of $\cos\theta_1$, $\cos\theta_2$, $\cos\theta_3$ in terms of the sides of the triangles OAP, OBP, OCP respectively, we have

$$\sin a_1 \left\{\frac{\cos\delta_1 - \cos d_1 \cos OP}{\sin d_1 \sin OP}\right\} + \sin a_2 \left\{\frac{\cos\delta_2 - \cos d_2 \cos OP}{\sin d_2 \sin OP}\right\}$$
$$+ \sin a_3 \left\{\frac{\cos\delta_3 - \cos d_3 \cos OP}{\sin d_3 \sin OP}\right\} = 0;$$

hence,

$$\cos OP = \frac{\sin a_1 \sin d_2 \sin d_3 \cos\delta_1 + \sin a_2 \sin d_1 \sin d_3 \cos\delta_2 + \sin a_3 \sin d_1 \sin d_2 \cos\delta_3}{\sin a_1 \sin d_2 \sin d_3 \cos d_1 + \sin a_2 \sin d_1 \sin d_3 \cos d_2 + \sin a_3 \sin d_1 \sin d_2 \cos d_3}$$
$$= \frac{\sin a_1 \operatorname{cosec} d_1 \cos\delta_1 + \sin a_2 \operatorname{cosec} d_2 \cos\delta_2 + \sin a_3 \operatorname{cosec} d_3 \cos\delta_3}{\sin a_1 \cot d_1 + \sin a_2 \cot d_2 + \sin a_3 \cot d_3} \ldots\ldots (b)$$

Draw perpendiculars OD, OE, OF from O upon the sides BC, CA, AB, and denote them by λ_1, λ_2, λ_3 respectively: then by the usual expressions in spherics, we have

$$\sin d_2 \sin d_3 \sin a_1 = \sin a \sin\lambda_1$$
$$\sin d_1 \sin d_3 \sin a_2 = \sin b \sin\lambda_2$$
$$\sin d_1 \sin d_2 \sin a_3 = \sin c \sin\lambda_3;$$

a, b, c denoting the sides BC, CA, AB respectively; then eq. (b) gives

$$\cos OP = \frac{\sin a \sin\lambda_1 \cos\delta_1 + \sin b \sin\lambda_2 \cos\delta_2 + \sin c \sin\lambda_3 \cos\delta_3}{\sin a \sin\lambda_1 \cos d_1 + \sin b \sin\lambda_2 \cos d_2 + \sin c \sin\lambda_3 \cos d_3} \ldots\ldots (c)^*$$

THEOREM I.

Let r be the radius of a circle passing through the points of contact of any three circles in mutual contact on a sphere, and ρ the radius of a circle touching all three; then if O, P, be the centres of these two circles, the distance between them will be obtained from the equation

$$\cos OP = \cos r \cos\rho \pm 2 \sin r \sin\rho;$$

the upper sign taking place in the double sign $\pm$ when the circle radius ρ is situated within the circle radius r, and the lower sign when the former circle is situated without the latter one.

For let A, B, C represent the centres, and ρ_1, ρ_2, ρ_3 the radii of three circles in mutual contact on a sphere; and let O be the centre of the circle, radius r, passing through their points of contact, and P the centre of a circle, radius ρ, touching the same three externally. Then in this case

$$\lambda_1 = \lambda_2 = \lambda_3 = r;\ \delta_1 = \rho + \rho_1;\ \delta_2 = \rho + \rho_2;\ \delta_3 = \rho + \rho_3;$$
$$a = \rho_2 + \rho_3;\ b = \rho_1 + \rho_3;\ c = \rho_1 + \rho_2;$$
$$\cos d_1 = \cos r \cos\rho_1;\ \cos d_2 = \cos r \cos\rho_2;\ \cos d_3 = \cos r \cos\rho_3.$$

* This equation shows that if A, B, C be three given points on a sphere, and the distance of any other point P on this sphere from the three given ones be denoted by δ_1, δ_2, δ_3 respectively; then if A_1, A_2, A_3 be three given quantities, and

$$A_1 \cos\delta_1 + A_2 \cos\delta_2 + A_3 \cos\delta_3 = \text{a constant quantity},$$

the locus of P will be a circle.

Hence substituting these values in (c) we have

$$\cos OP=\frac{\sin(\rho_2+\rho_3)\cos(\rho+\rho_1)+\sin(\rho_1+\rho_3)\cos(\rho+\rho_2)+\sin(\rho_1+\rho_2)\cos(\rho+\rho_3)}{\cos r\{\sin(\rho_2+\rho_3)\cos\rho_1+\sin(\rho_1+\rho_3)\cos\rho_2+\sin(\rho_1+\rho_2)\cos\rho_3\}}$$

$$=\frac{\sin\rho}{\cos r}\cdot\frac{\cot\rho(\cot\rho_1\cot\rho_2+\cot\rho_2\cot\rho_3+\cot\rho_3\cot\rho_1)-(\cot\rho_1+\cot\rho_2+\cot\rho_3)}{\cot\rho_1\cot\rho_2+\cot\rho_2\cot\rho_3+\cot\rho_3\cot\rho_1}$$

$$=\frac{\sin\rho}{\cos r}\cdot\frac{\cot\rho\operatorname{cosec}^2 r-\cot\rho\pm 2\cot r}{\operatorname{cosec}^2 r},$$ Math., No. II. pp. 101, 102, eq. (1, 8.)

$$=\frac{\sin\rho}{\cos r}\cdot\frac{\cot\rho\cot^2 r\pm 2\cot r}{\operatorname{cosec}^2 r}=\cos\rho\cos r\pm 2\sin\rho\sin r.$$

THEOREM II.

Let O, P be the centres and ρ, ρ' the radii of two circles respectively touching the same three circles in mutual contact on a sphere; then

$$\cos OP=\cos\rho\cos\rho'-7\sin\rho\sin\rho'.$$

Let A, B, C represent the centres, and ρ_1, ρ_2, ρ_3 the radii of the three circles in contact, and O, P the centres of the two circles radii ρ, ρ' respectively touching all three*; also let r, r_1, r_2, r_3 be the radii of the four circles that may be described to touch each other mutually in the same points as the four circles $\rho, \rho_1, \rho_2, \rho_3$ have contact in.

Then,

$$d_1=\rho+\rho_1;\quad d_2=\rho+\rho_2;\quad d_3=\rho+\rho_3,$$
$$\delta_1=\rho'+\rho_1;\quad \delta_2=\rho'+\rho_2;\quad \delta_3=\rho'+\rho_3;$$

and since the circles radii r_1, r_2, r_3 are respectively the inscribed ones of the triangles BOC, COA, AOB,† we shall have, from the common expression for the radius of a circle inscribed in a spherical triangle,

$$\sin d_2\sin d_3\sin a_1=2\cot r_1\sin\frac{d_2+d_3-a}{2}\sin\frac{a+d_2-d_3}{2}\sin\frac{a+d_3-d_2}{2}$$
$$=2\cot r_1\sin\rho\sin\rho_2\sin\rho_3;$$
$$\sin d_1\sin d_3\sin a_2=2\cot r_2\sin\rho\sin\rho_1\sin\rho_3;$$
$$\sin d_1\sin d_2\sin a_3=2\cot r_3\sin\rho\sin\rho_1\sin\rho_2;$$

Hence, substituting these in eq. (b), we have

$$\cos OP=\frac{\cot r_1\sin\rho_2\sin\rho_3\cos(\rho'+\rho_1)+\cot r_2\sin\rho_1\sin\rho_3\cos(\rho'+\rho_2)+\cot r_3\sin\rho_1\sin\rho_2\cos(\rho'+\rho_3)}{\cot r_1\sin\rho_2\sin\rho_3\cos(\rho+\rho_1)+\cot r_2\sin\rho_1\sin\rho_3\cos(\rho+\rho_2)+\cot r_3\sin\rho_1\sin\rho_2\cos(\rho+\rho_3)}$$

$$=\frac{\sin\rho'}{\sin\rho}\cdot\frac{\cot r_1(\cot\rho'\cot\rho_1-1)+\cot r_2(\cot\rho'\cot\rho_2-1)+\cot r_3(\cot\rho'\cot\rho_3-1)}{\cot r_1(\cot\rho\cot\rho_1-1)+\cot r_2(\cot\rho\cot\rho_2-1)+\cot r_3(\cot\rho\cot\rho_3-1)}$$

$$=\frac{\sin\rho'}{\sin\rho}\cdot\frac{\cot\rho'(\cot\rho_1\cot r_1+\cot\rho_2\cot r_2+\cot\rho_3\cot r_3)-(\cot r_1+\cot r_2+\cot r_3)}{\cot\rho(\cot\rho_1\cot r_1+\cot\rho_2\cot r_2+\cot r_3\cot\rho_3)-(\cot r_1+\cot r_2+\cot r_3)}\quad\ldots.(1)$$

And from the properties of circles in mutual contact on a sphere, eq. (5), Math. No. II., p. 102, we have

* The circle radius ρ being considered the less of the two circles of contact, throughout the demonstration.

† Properties of circles in mutual contact on a sphere, Mathematician, No. II. p. 101.

$$2(\cot\rho_1\cot r_1+\cot\rho_2\cot r_2+\cot\rho_3\cot r_3)=\cot\rho_1(\cot\rho-\cot\rho_1+\cot\rho_2+\cot\rho_3)$$
$$+\cot\rho_2(\cot\rho+\cot\rho_1-\cot\rho_2+\cot\rho_3)+\cot\rho_3(\cot\rho+\cot\rho_1+\cot\rho_2-\cot\rho_3)$$
$$=(\cot\rho-\cot_1-\cot\rho_2-\cot\rho_3)(\cot\rho_1+\cot\rho_2+\cot\rho_3)$$
$$+4(\cot\rho_1\cot\rho_2+\cot\rho_2\cot\rho_3+\cot\rho_3\cot\rho_1)$$
$$=2\cot r(\cot\rho-2\cot r)+4\cot^2 r+4;$$

Ibid, pp. 101, 102, eqs. (1), (8).

$$\therefore\ \cot\rho_1\cot r_1+\cot\rho_2\cot r_2+\cot\rho_3\cot r_3=2+\cot r\cot\rho\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Also, $\cot r_1+\cot r_2+\cot r_3=2\cot\rho-\cot r$; *Ibid*, p. 102, eq. 9.

Substituting these last two equations in (1), we have

$$\cos\mathrm{OP}=\frac{\sin\rho'}{\sin\rho}\cdot\frac{\cot\rho'(2+\cot r\cot\rho)+\cot r-2\cot\rho}{\cot\rho\,(2+\cot r\cot\rho)+\cot r-2\cot\rho}$$
$$=\frac{\sin\rho'\,\sin\rho}{\cot r}\,\{2(\cot\rho'-\cot\rho)+\cot r(1+\cot\rho'\cot\rho)\}\ \ldots\ldots(3)$$

But $\cot\rho=\cot\rho_1+\cot\rho_2+\cot\rho_3+2\cot r$; $\cot\rho'=\cot\rho_1+\cot\rho_2+\cot\rho_3-2\cot r$;

Ibid, p. 102, eq. (8).

$$\therefore\quad \cot\rho-\cot\rho'=4\cot r\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Substituting (4) in (3), gives

$$\cos\mathrm{OP}=\sin\rho_{,}\sin\rho\{-8+(1+\cot\rho'\cot\rho)\}$$
$$=\cos\rho\cos\rho'-7\sin\rho\sin\rho'\ldots\ldots\ldots\ldots\ldots\ldots(d)$$

Obs. In the preceding investigation the two circles of contact have been considered as touched externally by the three given circles (radii ρ_1, ρ_2, ρ_3); the result (d) will, however, be general for every position in which they may be touched by the given three, by considering the radius of either of these two circles a negative quantity in this expression, when it is touched exernally by the given ones.

THEOREM III.

Let O be the centre of the inscribed circle of a spherical triangle ABC, and r its radius; and let p_1, p_2, p_3 denote the perpendiculars from the angles A, B, C upon the opposite sides: then if δ_1, δ_2, δ_3 denote the spherical distances of any other point P, on the sphere, from the angles A,B,C respectively, we shall have

$$\frac{\cos\mathrm{OP}}{\sin r}=\frac{\cos\delta_1}{\sin p_1}+\frac{\cos\delta_2}{\sin p_2}+\frac{\cos\delta_3}{\sin p_3}.$$

For, applying equation (c), we have in this case, $\lambda_1=\lambda_2=\lambda_3=r$,

$\cos d_1=\cos r\cos(s-a)$; $\cos d_2=\cos r\cos(s-b)$; $\cos d_3=\cos r\cos(s-c)$;

and substituting these in (c), we have

$$\cos\mathrm{OP}=\frac{\sin a\cos\delta_1+\sin b\cos\delta_2+\sin c\cos\delta_3}{\cos r\{\sin a\cos(s-a)+\sin b\cos(s-b)+\sin c\cos(s-c)\}}\ldots(5)$$

But,

$$\left.\begin{aligned}\sin a&=\sin\{(s-b)+(s-c)\}=\sin(s-b)\cos(s-c)+\cos(s-b)\sin(s-c)\\ \sin b&=\sin\{(s-a)+(s-c)\}=\sin(s-a)\cos(s-c)+\cos(s-a)\sin(s-c)\\ \sin c&=\sin\{(s-a)+(s-b)\}=\sin(s-a)\cos(s-b)+\cos(s-a)\sin(s-b)\end{aligned}\right\}(6)$$

$$\therefore\ \sin a\cos(s-a)+\sin b\cos(s-b)+\sin c\cos(s-c)$$
$$=2\left\{\begin{aligned}&\sin(s-b)\cos(s-c)\cos(s-a)+\sin(s-c)\cos(s-b)\cos(s-a)\\&+\sin(s-a)\cos(s-b)\cos(s-c)\end{aligned}\right\}$$

$$=2\{\sin(s-a+s-b+s-c)+\sin(s-a)\sin(s-b)\sin(s-c)\}$$
$$=2\sin s+2\sin(s-a)\sin(s-b)\sin(s-c)$$
$$=2\sin s+2\sin s\tan^2 r^{*}=\frac{2\sin s}{\cos^2 r}$$
$$=\frac{2n}{\sin r\cos r}=\frac{4n}{\sin 2r}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Hence (5) becomes

$$\cos \mathrm{OP}=\frac{\sin r}{2n}(\sin a\cos\delta_1+\sin b\cos\delta_2+\sin c\cos\delta_3)\ldots\ldots\ldots\ldots\ldots\ldots(8)$$
$$=\sin r\left\{\frac{\cos\delta_1}{\sin p_1}+\frac{\cos\delta_2}{\sin p_2}+\frac{\cos\delta_3}{\sin p_3}\right\};\ \text{since}\ \sin p_1=\frac{2n}{\sin a},\ etc.$$
$$\therefore\ \frac{\cos\mathrm{OP}}{\sin r}=\frac{\cos\delta_1}{\sin p_1}+\frac{\cos\delta_2}{\sin p_2}+\frac{\sin\delta_3}{\sin p_3}.$$

Let P coincide with the centre of the circle circumscribing the triangle ABC; then $\delta_1=\delta_2=\delta_3=\mathrm{R}$, and from (8) we have

$$\cos\mathrm{OP}=\frac{\sin r\cos\mathrm{R}}{2n}(\sin a+\sin b+\sin c)$$
$$=\sin r\cos\mathrm{R}(\operatorname{cosec}p_1+\operatorname{cosec}p_2+\operatorname{cosec}p_3)$$
$$=\frac{\cos\mathrm{R}\cos r}{2\sin s}(\sin a+\sin b+\sin c).$$

Let O_1, O_2, O_3 be the centres, and r_1, r_2, r_3 the radii of the escribed circles; then

$$\left.\begin{aligned}\frac{\cos\mathrm{O}_1\mathrm{P}}{\sin r_1}&=-\frac{\cos\delta_1}{\sin p_1}+\frac{\cos\delta_2}{\sin p_2}+\frac{\cos\delta_3}{\sin p_3}\\ \frac{\cos\mathrm{O}_2\mathrm{P}}{\sin r_2}&=\frac{\cos\delta_1}{\sin p_1}-\frac{\cos\delta_2}{\sin p_2}+\frac{\cos\delta_3}{\sin p_3}\\ \frac{\cos\mathrm{O}_3\mathrm{P}}{\sin r_3}&=\frac{\cos\delta_1}{\sin p_1}+\frac{\cos\delta_2}{\sin p_2}-\frac{\cos\delta_3}{\sin p_3}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(e)$$

For let O, in the figure to the general theorem (A), coincide with O_1; then $-\lambda_1=\lambda_2=\lambda_3=r_1$,

$\cos d_1=\cos r_1\cos s$; $\cos d_2=\cos r_1\cos(s-c)$; $\cos d_3=\cos r_1\cos(s-b)$;

and substituting these in (c) and pursuing a process of reduction similar to the above, the first of the equations (e) is established. In a similar way the others are established.

Cor. 1. $\dfrac{2\cos\delta_1}{\sin p_1}=\dfrac{\cos\mathrm{O}_2\mathrm{P}}{\sin r_2}+\dfrac{\cos\mathrm{O}_3\mathrm{P}}{\sin r_3}$; $\dfrac{2\cos\delta_2}{\sin p_2}=\dfrac{\cos\mathrm{O}_1\mathrm{P}}{\sin r_1}+\dfrac{\cos\mathrm{O}_3\mathrm{P}}{\sin r_3}$;

$$\frac{2\cos\delta_3}{\sin p_3}=\frac{\cos\mathrm{O}_1\mathrm{P}}{\sin r_1}+\frac{\cos\mathrm{O}_2\mathrm{P}}{\sin r_2}.$$

Cor. 2. $\dfrac{\cos\mathrm{O}_1\mathrm{P}}{\sin r_1}+\dfrac{\cos\mathrm{O}_2\mathrm{P}}{\sin r_2}+\dfrac{\cos\mathrm{O}_3\mathrm{P}}{\sin r_3}=\dfrac{\cos\delta_1}{\sin p_1}+\dfrac{\cos\delta_2}{\sin p_2}+\dfrac{\cos\delta_3}{\sin p_3}=\dfrac{\cos\mathrm{OP}}{\sin r}$.

Scholium. Many remarkable properties may be deduced from the general

* From the usual expression for the radius of a circle touching the sides of a spherical triangle.

equations in the last two theorems, and we might also express the distances of any point from the centres of the circles circumscribing the "Associated System of Spherical Triangles," in neat forms, by means of the general theorem: but sufficient illustration of the application of that theorem has been given. From the preceding properties, we shall now obtain their analogous ones on a plane, and then deduce from (a) a general theorem for determining the distances between any two points on a plane, of which one is fixed and the other variable in position, in terms of the distances of the variable point from the three other given ones on the plane, and of the triangles formed by joining the four given points.

If an arc be indefinitely diminished in respect to its radius, its cosine ultimately becomes equal to the radius, and its sine then equals the arc itself; hence the above properties afford the well known expression:

$$\frac{1}{r}=\frac{1}{p_2}+\frac{1}{p_2}+\frac{1}{p_3};\ \frac{1}{r_1}=-\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3};\ \frac{1}{r_2}=\frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{p_3},\ etc.$$

$$\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3},\ \text{by theor. III, cor. 2.}$$

THEOREM IV.

Let ρ, P denote the radius and centre respectively of a great circle touching three given ones in mutual contact on a plane, and r, O the radius and centre respectively of a circle passing through their points of contact; then

$$OP^2 = r^2+\rho^2 \pm 4r\rho;$$

the upper sign taking place when the centre of the lesser circle is situate without the greater circle, and the lower sign, when situate within it.

For taking the analogous property on the sphere in Theorem I,

$$\cos OP = \cos r \cos \rho \pm 2\sin r \sin \rho.$$

Let R denote the radius of the sphere on which the circles are described; then

$$\cos OP = 1 - \frac{OP^2}{1.2R^2} + \frac{OP^4}{1\cdot2\cdot3\cdot4R^4} - \dots\dots\dots\dots (1)$$

$$\therefore \cos r = 1 - \frac{r^2}{1\cdot2R^2} - \frac{r^4}{1\cdot2\cdot3\cdot4R^4} - \dots \qquad \cos\rho = 1 - \frac{\rho^2}{1\cdot2R^2} + \frac{\rho^4}{1\cdot2\cdot3\cdot4R^4} - \dots$$

$$\sin r = \frac{r}{R} - \frac{r^3}{1\cdot2\cdot3R^3} + \dots\dots \qquad \sin\rho = \frac{\rho}{R} - \frac{\rho^3}{1\cdot2\cdot3R^3} - \dots\dots$$

$$\text{Hence } \cos r \cos\rho \pm 2\sin r\sin\rho = \left\{ 1 - \frac{r^2+\rho^2}{2R^2} + \frac{r^4+\rho^4+6\rho^2r^2}{24R^4} - \dots \right\}$$

$$\pm \left\{ \frac{2\rho r}{R^2} - \frac{\rho r(r^2+\rho^2)}{3R^4} + \dots \right\};\ \text{and equating this with (1),}$$

$$r^2+\rho^2 \pm 4\rho r - \frac{r^4+\rho^4 \pm 4\rho r(r^2+\rho^2)}{12R^2} + \dots = OP^2 - \frac{OP^4}{12R^2} + \dots;$$

which, when R becomes infinite, or the sphere becomes a plane, gives

$$r^2+\rho^2 \pm 4r\rho = OP^2.$$

The double sign $\mp$ being treated in this form exactly as its inverted form $\pm$ is treated in Theorem I.

THEOREM V.

Let ρ', ρ denote the radii of two circles respectively touching three others in mutual contact on a sphere, and let O, P be their centres: then

$$OP^2 = \rho'^2 + \rho^2 \pm 14\rho'\rho;$$

the upper sign $+$ applying when both the circles are touched externally by the given three, and the sign $-$, when one of the two is touched internally by the given three.

The analogous property on the sphere as given in Theorem II, is

$$\cos OP = \cos\rho\cos\rho' \pm 7\sin\rho\sin\rho';$$

and substituting the values of cos OP, $\cos\rho$, $\cos\rho'$, $\sin\rho$, $\sin\rho'$, as in the preceding example, we have

$$1 - \frac{OP^2}{1\cdot2R^2} + \frac{OP^4}{1\cdot2\cdot3\cdot4R^4} - \ldots = \left(1 - \frac{\rho^2}{2R^2} + \frac{\rho^4}{24R^4} - \ldots\right)\left(1 - \frac{\rho'^2}{2R^2} + \frac{\rho'^4}{24R^4} - \ldots\right)$$
$$\pm 7\left(\frac{\rho}{R} - \frac{\rho^3}{6R^3} + \ldots\right)\left(\frac{\rho'}{R} - \frac{\rho'^3}{6R^3} + \ldots\right)$$

Or, multiplying out and cancelling on both sides the resulting equation, we get

$$OP^2 - \frac{OP^4}{12R^2} + \ldots = \rho^2 + \rho'^2 \mp 14\rho\rho' - \frac{\rho'^4 + \rho^4 + 6\rho^2\rho'^2 \mp 28\rho\rho'(\rho^2 + \rho'^2)}{12R^2} + \ldots$$

And when $R = \infty$, or the sphere becomes a plane, we have

$$OP^2 = \rho^2 + \rho'^2 \pm 14\rho\rho'.$$

ON A PRINCIPLE IN THE THEORY OF PROBABILITIES.

[*From a Correspondent.*]

Let $p_1, p_2, p_3, \ldots p_n$ be the respective probabilities of happening of n independent events: then the following general principle will have place, viz.

$p_1 + p_2 + p_3 + \ldots p_n =$ the prob. of one of the events, *at least*, happening,
$+$ prob. of two at least happening,
$+$ prob. of three at least happening,
:
$+$ prob. of all happening together.

This principle has not, I believe, hitherto been given by any writer on Probabilities. It is interesting as affording an intelligible interpretation of the sum of the probabilities of any number of independent events; and is moreover useful in enabling us very readily to determine certain compound probabilities when others are already known.

Thus:—let there be but two events; then by the preceding principle

$p_1 + p_2 =$ prob. of one at least happening $+$ prob. of both happening.

But the probability of both happening is known to be $p_1 p_2$,

$\therefore p_1 + p_2 - p_1 p_2 =$ prob. of one at least happening.

Again :—let there be three events ; then replacing p_1, p_2, p_3, by the cc binations p_1p_2, p_1p_3, p_2p_3, we have, by the same principle,

$p_1p_2+p_1p_3+p_2p_3$=prob. of one of these compound events at least happeni
+prob. of two at least,
+prob. of all the compound events.

But two of the compound events, at least, is obviously the same as all t three simple events; and so of course, is the conjunction of all the co pound events; and the probability of all the simple events happening known to be $p_1p_2p_3$: hence

$p_1p_2+p_1p_3+p_2p_3$= prob. of two of the events at least happening togetl
$+2p_1p_2p_3$;

$\therefore p_1p_2+p_1p_3+p_2p_3-2p_1p_2p_3$=prob. of two of the events at least happen

Moreover, the probability of one of the events at least happening is, the principle, equal to the sum of the individual probabilities, diminished the expression just deduced, and by $p_1p_2p_3$: that is,

$p_1+p_2+p_3-p_1p_2-p_1p_3-p_2p_3+p_1p_2p_3$ = prob. of one, at least, of t three events happening. And so on.

Belfast, Sept. 1844.

P.S. At the bottom of page 122, last number, the following should added :—

From either of the above forms the other may be conveniently deduc by aid of the fundamental relation $\sin^2+\cos^2=1$: thus, taking the last those forms, we have

$$\sin^2(a\pm b)=1-\cos^2(a\pm b)=1-\left(\frac{1\mp\tan a\tan b}{\sec a\sec b}\right)^2=\left(\frac{\tan a\pm\tan b}{\sec a\sec b}\right)^2.$$

ON THE TRANSFORMATION OF ALGEBRAIC EQUATION

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1. The object of this paper is the consideration of certain matters of det arising, directly or indirectly, out of communications which have appeare under the above title, in the last two numbers of the *Mathematician.*

2. The method adopted by me for the purpose of finding the values $\pi(v)$, &c., was, to write the coefficients of $\phi_1(v)$ in a horizontal row; unc this row to write those of $\phi_2(v)$; then, in order to determine the coeffici of any given term of the product, I formed, by inspection, all the com nations of the coefficients corresponding to such of the quantities v_1, v_2, & as entered into that term, subject to the condition, that no two of the coe cients should be in the same horizontal line. But, as a direct method m be thought more satisfactory, I shall here calculate the value of $\pi(v)$ biquadratics; the operation is not long, although the three factors ea consists of four terms; for cubics it is so simple that it is unnecessary notice it here, or to point out the *practical* value of the function π in the transformations. I would, however, observe that a mechanical mode finding the product of a number of factors, each consisting of several tern on which I have lately been engaged, (the principle of which consists forming the various combinations of the terms of these factors,) is peculia

applicable to expressions of the form $\phi(v)$, for, if we represent any one of their terms, $a^r v_m$ for instance, by r_m, the coefficient of the product of any number of these terms will be found by *adding* the coefficients of each term.

3. Now, to find $\pi(v)$ when $n = 4$, let

$a = v_1 - v_2$, $b = v_3 - v_4$, $c = v_1 + v_2$, $d = v_3 + v_4$, then

$\phi_1(v) = a + b$, $\phi_2(v) = a - b$, $\phi_3(v) = c - d$, and

$\pi(v) = (a+b)(a-b)(c-d) = a^2c + b^2d - a^2d - b^2c$, but

$a^2c = (v_1 - v_2)(v_1^2 - v_2^2) = v_1^3 + v_2^3 - v_1^2v_2 - v_2^2v_1$, so

$b^2d = v_3^3 + v_4^3 - v_3^2v_4 - v_4^2v_3$, and

$a^2d = (v_1^2 + v_2^2)(v_3 + v_4) - 2v_1v_2(v_3 + v_4)$, so

$b^2c = (v_3^2 + v_4^2)(v_1 + v_2) - 2v_3v_4(v_1 + v_2)$,

the second and fourth lines being, respectively, formed from the first and third, by changing v_1 and v_2, respectively, into v_3 and v_4, and *vice versa*. Subtract the sum of the last two lines from that of the first two, and we have

$$\pi(v) = \Sigma(v_1^3) - \Sigma(v_1^2v_2) + 2\Sigma(v_1v_2v_3), \text{ as at page 83.}$$

4. On changing v_1, &c., into x_1^λ, &c., we obtain $\pi(x^\lambda)$, and we might determine $\pi'(x^\lambda)$ by a similar process, but this is unnecessary, since $\pi'(x^\lambda)$, $\pi''(x^\lambda)$, &c., are connected with $\pi(x^\lambda)$, and with one another, in a manner pointed out in my first communication, and which, though its discussion does not fall within the purpose of this paper, will be investigated at another time.

5. By way of supplement to the first paper, I may remark, that when $n = 1$, $\pi(v)$ is identically zero, as we might expect; that $\Sigma(\phi(v) = (n-1)v_1 - \Sigma(v_2)$ for which expression, when $n = 3$, see Murphy's Equations, p. 53; that, in the developement of the equation (7) of that paper,

$$\pi'(x^\lambda) = \Sigma(x^{2\lambda+\lambda'}) - \Sigma(x_1^{2\lambda}x_2^{\lambda'}) - \Sigma(x_1^{\lambda+\lambda'}x_2^{\lambda}) + 2\Sigma(x_1^\lambda x_2^\lambda x_3^{\lambda'});$$

that the interchange of λ and λ', in the above expression, gives $\pi''(x^\lambda)$, while $\pi'''(x^\lambda)$ is obviously derived from $\pi(v)$ by changing v into $x^{\lambda'}$; and finally, that the simple equations, of which the last equation at p. 34 is the product, are

$$z\phi_1(x) + a^2\phi_2(x_1x_2) = 0, \text{ and } z\phi_2(x) + a\phi_1(x_1x_2) = 0.$$

6. Next, let z_1 and z_2 be the values of z in the quadratic last alluded to, and which is equivalent to

$$(a^2 - 3b)z^2 + (ab - 9c)z + (b^2 - 3ac) = 0;$$

also, let $\rho_1 = a + 3z_1$, and $\rho_2 = a + 3z_2$, then

$$\rho_1\rho_2 = a^2 + 3a(z_1 + z_2) + 9z_1z_2$$

$$= a^2 - 3a \cdot \frac{ab - 9c}{a^2 - 3b} + 9 \cdot \frac{b^2 - 3ac}{a^2 - 3b}$$

$$= \frac{1}{n}\{a^4 - 6a^2b + 9b^2\} \text{ where } n = a^2 - 3b$$

$$= n, \text{ hence } \frac{n^2}{\rho_1} = n\rho_2:$$

but, my expression for x, at p. 249, vol. ii. of the Cam. Math. Journ., is equivalent to $x = -\frac{1}{3}\{a + (1)^{\frac{1}{3}}\sqrt[3]{n\rho} + (1)^{\frac{2}{3}}\sqrt[3]{\frac{n^2}{\rho}}\}$ whence

$$x = -\tfrac{1}{3}\{a + (1)^{\frac{1}{3}}\sqrt[3]{n\rho_1} + (1)^{\frac{2}{3}}\sqrt[3]{n\rho_2}\}.$$

7. The following rule will impress the above on the memory, and will be found useful whenever we have to determine the *exact* expression for the roots of a cubic. Its correctness will be seen by examining the symbolical example; the numerical one is taken from Mr. Davies's edition of Hutton's Course. vol. i., p. 200, ex. 2. A rather singular application of the rule will be found in the last example given by Mr. Hind (in his *Algebra,*) in illustration of the solution on which it is founded; there, though all the roots are real, the given equation is withdrawn from the irreducible case*, in consequence of its coefficients satisfying the relation

$$\left(\frac{a}{3}\right)^3 - \frac{a.b}{3.2} + \frac{c}{2} = 0.$$

8. By the operation (1) I mean multiplying a quantity by a, and subtracting from the product three times another quantity. In the first step of the following process, the quantities to be subtracted are those at the top of the columns next on the right of the one in which we are operating; in the last, they are those which stand at the bottom. This being premised, and a, b, c being the coefficients of the given cubic,

(a.) Place the quantities a, b, and $3c$ in a horizontal line; multiply the right hand quantity by the left, and subtract from the product the square of the middle one;† perform [1] on the middle and left hand quantities, and divide the middle result by 2.

(b.) Of the three new quantities at the bottom of the columns, multiply the extremes, as before, but *add* to the product the square of the mean; find the square roots of the result, and add them separately to the (new) middle quantity; perform [1] on the (new) left hand quantity, using these last sums successively; add the cube roots of the results to a, and divide the sum by 3; the quotient with its sign changed is the value of x.

Ex. I.	a	b	$3c$
	a	a	
	a^2	ab	$3ac$
	$3b$	$9c$	b^2
		$2)2b'$	

* The curious on this subject may adapt Mr. Barlow's table for the irreducible case to the rule, by considering it as a register of the values of twice the real part of the cube root of the expression $a+(.\dot{0}3\dot{7}-a^2)^{\frac{1}{2}}\sqrt{-1}$, from $a = 0$ to $a = .192455$.

† In the few cases in which the remainder is zero, perform [1] on a, multiply the result by a, extract the cube root, and subtract a from it, then b, divided by this difference, will give the real root, the others being imaginary.

	a'	b'		c'
				a'
				$a'c'$
				b'^2
		$\pm d$		d^2
	a	$e \times 3 = 3e$		
	aa'	f	aa'	θ'
	$3f$		θ'^3	...... θ
	θ^2			a
				$3)g$
				$-x$
Ex. II.	-7	14		-60
	-7	-7		-7
	49	-98		420
	42	-180		196
	7	$2)82$		224
		41		7
				1568
	-7	± 57		1681
	-49	$98 \times 3 =$	294	3249
	-48	-16	-49	
	-1		-343	.... -7
				-1
				-7
				$3)-15$
				$-5 \therefore x=5.$

When d^2, the quantity at the bottom of the right hand column, is positive, equation has one real root; when negative, it has three. In the latter we obtain, by the above rule, the value of x under an imaginary form, ich we, however, arrive at much sooner than by the algebraical process.

Temple, 1844

ON DEFINITE INTEGRALS.

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This paper exhibits an easy method of finding definite integrals; for which urpose the following subsidiary formulæ are required.

In $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots$ change x into $xe^{\theta\sqrt{-1}}$ and $xe^{-\theta\sqrt{-1}}$,

and add results; remembering that $\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$, we find

$\cos 5\theta - \ldots\ldots$

Similarly, by changing x into $\frac{x}{\sqrt{-1}}e^{\theta\sqrt{-1}}$ and $\frac{x}{\sqrt{-1}}e^{-\theta\sqrt{-1}}$, and subtracting results; we find

$$\tfrac{1}{2}\tan^{-1}\left(\frac{2x\sin\theta}{1-x^2}\right)=x\sin\theta+\frac{x^3}{3}\sin 3\theta+\frac{x^5}{5}\sin 5\theta+\ldots\ldots\ldots(b)$$

In like manner, from $\log\left(\frac{1}{1-x}\right)=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots$ we obtain

$$-\tfrac{1}{2}\log(1-2x\cos\theta+x^2)=x\cos\theta+\frac{x^2}{2}\cos 2\theta+\frac{x^3}{3}\cos 3\theta+\ldots(c)$$

$$\tfrac{1}{2}\log(1+2x\cos\theta+x^2)=x\cos\theta-\frac{x^2}{2}\cos 2\theta+\frac{x^3}{3}\cos 3\theta-\ldots(d)$$

$$\tfrac{1}{4}\log\left(\frac{1+2x\cos\theta+x^2}{1-2x\cos\theta+x^2}\right)=x\cos\theta+\frac{x^3}{3}\cos 3\theta+\frac{x^5}{5}\cos 5\theta+\ldots(e)$$

$$\tfrac{1}{4}\log\left(\frac{1+2x\sin\theta+x^2}{1-2x\sin\theta+x^2}\right)=x\sin\theta-\frac{x^3}{3}\sin 3\theta+\frac{x^5}{5}\sin 5\theta-\ldots(f)$$

The last is derived from the preceding one by changing θ into $\frac{\pi}{2}-\theta$. We easily find the two following by the aid of the known formula

$$A=\frac{1}{2\sqrt{-1}}\log\left(\frac{1+\tan A\sqrt{-1}}{1-\tan A\sqrt{-1}}\right).$$

$$\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right)=x\sin\theta+\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta+\ldots\ldots\ldots(g)$$

$$\tan^{-1}\left(\frac{x\sin\theta}{1+x\cos\theta}\right)=x\sin\theta-\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta-\ldots\ldots\ldots(h)$$

In the same manner from $\frac{x}{1-x}=x+x^2+x^3+\ldots$ we derive

$$\frac{x\cos\theta-x^2}{1-2x\cos\theta+x^2}=x\cos\theta+x^2\cos 2\theta+x^3\cos 3\theta+\ldots\ldots\ldots\ldots(i)$$

$$\frac{x\sin\theta}{1-2x\cos\theta+x^2}=x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\ldots\ldots\ldots\ldots(k)$$

And as before, $e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2\,3}+\ldots$ will give

$$e^{x\cos\theta}\cos(x\sin\theta)=1+x\cos\theta+\frac{x^2}{2}\cos 2\theta+\frac{x^3}{2\cdot 3}\cos 3\theta+\ldots(l)$$

$$e^{x\cos\theta}\sin(x\sin\theta)=x\sin\theta+\frac{x^2}{2}\sin 2\theta+\frac{x^3}{2\cdot 3}\sin 3\theta+\ldots\ldots(m)$$

The integrals following are taken from $x=0$ to ∞. To begin with $\int\frac{dx}{x}\sin rx=\frac{\pi}{2}$, which is well known; make $r=1, 3, 5,$ *etc.*, and multiply the results by h, $\frac{h^3}{3}$, $\frac{h^5}{5}$, *etc.*, respectively, and take the sum to infinity.

That of the first member is given by (b), that of the second is easily found. Thus we have

$$\int \frac{dx}{x}\tan^{-1}(a\sin x) = \frac{\pi}{2}\log\left(\frac{1+h}{1-h}\right),\ a = \frac{2h}{1-h^2}\ \ldots\ldots\ldots (1)$$

Again, make $r = 1, 2, 3,$ *etc.*; multiply by $h, h^2, h^3,$ *etc.*, and sum by (k): there results

$$\int \frac{dx}{x}\ \frac{\sin x}{1-2h\cos x+h^2} = \frac{\pi}{2}\cdot\frac{1}{1-h}\ \ldots\ldots\ldots (2)$$

Make $r=1, 3, 5,$ *etc.*; multiply by $h, -\frac{h^3}{3}, \frac{h^5}{5},$ *etc.*; and sum by (f):

$$\int \frac{dx}{x}\log\left(\frac{1+2h\sin x+h^2}{1-2h\sin x+h^2}\right) = 2\pi\tan^{-1}h \ldots\ldots\ldots (3)$$

Make $r=1, 2, 3,$ *etc.*; multiply by $h, -\frac{h^2}{2}, \frac{h^3}{3},$ *etc.*; and sum by (h):

$$\int \frac{dx}{x}\tan^{-1}\left(\frac{h\sin x}{1+h\cos x}\right) = \frac{\pi}{2}\log(1+h) \ldots\ldots\ldots (4)$$

In this last h must be a proper fraction, positive or negative.

Again, make $r=1, 2, 3,$ *etc.*; and multiply by $h, \frac{h^2}{2}, \frac{h^3}{2\cdot 3},$ *etc.*; and sum by (m):

$$\int \frac{dx}{x}\sin(h\sin x)\,e^{h\cos x} = \frac{\pi}{2}(e^{h}-1) \ldots\ldots\ldots (5)$$

Differentiate (1) for h, which gives

$$\int \frac{dx}{x}\ \frac{\sin x}{1+a^2\sin^2 x} = \frac{\pi}{2}\ \frac{1-h^2}{1+h^2} \ldots\ldots\ldots (6)$$

Let us now treat in the same manner the known integral

$$\int \frac{dx \cos rx}{m^2+x^2} = \frac{\pi}{2m}e^{-mr}.$$

Make $r=1, 2, 3,$ *etc.*; multiply by $h, \frac{h^2}{2}, \frac{h^3}{3},$ *etc.*; and sum by (c):

$$\int \frac{dx}{m^2+x^2}\log(1-2h\cos x+h^2) = \frac{\pi}{m}\log(1-he^{-m}) \ldots\ldots\ldots (7)$$

Make $r=1, 3, 5,$ *etc*; multiply by $h, -\frac{h^3}{3}, \frac{h^5}{5},$ *etc.*; and sum by (a):

$$\int \frac{dx}{m^2+x^2}\tan^{-1}(a\cos x) = \frac{\pi}{m}\tan^{-1}(he^{-m});\ \text{where } a = \frac{2h}{1-h^2} \ldots (8)$$

In the two last change m into $m\sqrt{-1}$ and $-m\sqrt{-1}$, and take half the sum of the results. Reducing by known formulæ, we obtain

$$\int \frac{dx}{x^2-m^2}\log(1-2h\cos x+h^2) = \frac{\pi}{m}\tan^{-1}\left(\frac{h\sin m}{1-h\cos m}\right) \ldots\ldots (9)$$

$$\int \frac{dx}{x^2-m^2}\tan^{-1}(a\cos x) = \frac{\pi}{4m}\log\left(\frac{1-2h\sin m+h^2}{1+2h\sin m+h^2}\right) \ldots\ldots (10)$$

Make $r=0, 1, 2, 3,$ *etc.*; multiply by $1, h, \frac{h^2}{2}, \frac{h^3}{2\cdot 3},$ *etc.*; and sum by (l):

$$\int \frac{dx}{m^2+x^2}\cos(h\sin x)e^{h\cos x} = \frac{\pi}{2m}e^{he^{-m}} \ldots\ldots\ldots\ldots\ldots\ldots (1$$

We now give some examples which require the sum of $\frac{1}{1^{2m}} \pm \frac{1}{2^{2m}} + \frac{1}{3^{2m}}$ $\pm \frac{1}{4^{2m}} + \ldots,$ which is known. In $\int dx e^{-rx} = \frac{1}{r}$, make $r=1, 2, 3,$ *etc.* multiply by $1, \frac{1}{2}, \frac{1}{3},$ *etc.*, and also by $1, -\frac{1}{2}, \frac{1}{3},$ *etc.*; and sum. We thus obtain

$$\int dx \log(1-e^{-x}) = -\frac{\pi^2}{6} \ldots (12) \quad \int dx \log(1+e^{-x}) = \frac{\pi^2}{12} \ldots (13)$$

In $\int x dx e^{-rx} = \frac{1}{r^2}$, $\int x^3 dx e^{-rx} = \frac{6}{r^4}$, make $r=1, 2, 3,$ *etc.*, and sum:

$$\int \frac{x dx}{e^x - 1} = \frac{\pi^2}{6}, \quad \int \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

In $\int e^{-ax} dx \sin rx = \frac{r}{a^2+r^2}$, $\int e^{-ax} dx \cos rx = \frac{a}{a^2+r^2}$, make $r=a=1,$ $2, 3,$ *etc.*; multiply by $1, \frac{1}{2}, \frac{1}{3},$ *etc.*, and sum by (g) and (d):

$$\int dx \tan^{-1}\left(\frac{\sin x}{e^x - \cos x}\right) = \frac{\pi^2}{12} \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

$$\int dx \log(1 - 2e^{-x}\cos x + e^{-2x}) = -\frac{\pi^2}{6} \ldots\ldots\ldots\ldots (16)$$

Make $a=p, 2p, 3p,$ *etc.*; $r=1, 2, 3,$ *etc.*; multiply by $1, \frac{1}{2}, \frac{1}{3},$ *etc.*; and sum as in the two last examples:

$$\int dx \tan^{-1}\left(\frac{\sin x}{e^{px} - \cos x}\right) = \frac{\pi^2}{6(p^2+1)} \ldots\ldots\ldots\ldots\ldots (17)$$

$$\int dx \log(1 - 2e^{-px}\cos x + e^{-2px}) = -\frac{p\pi^2}{3(p^2+1)} \ldots\ldots (18)$$

Make $a=1, 2, 3,$ *etc.*; $r=p, 2p, 3p,$ *etc.*; multiply by $1, \frac{1}{2}, \frac{1}{3},$ *etc*; and sum as before:

$$\int dx \tan^{-1}\left(\frac{\sin px}{e^x - \cos px}\right) = \frac{p\pi^2}{6(p^2+1)} \ldots\ldots\ldots\ldots\ldots (19)$$

$$\int dx \log(1 - 2e^{-x}\cos px + e^{-2x}) = -\frac{\pi^2}{3(p^2+1)} \ldots\ldots (20)$$

Make a and r as in the six preceding examples, but multiply by $1, -\frac{1}{2}, \frac{1}{3},$ *etc.*, and sum:

$$\int dx \tan^{-1}\left(\frac{\sin x}{e^x + \cos x}\right) = \frac{\pi^2}{24}; \quad \int dx \log(1 + 2e^{-x}\cos x + e^{-2x}) = \frac{\pi^2}{12} \ldots (21)$$

$$\left.\begin{aligned}\int dx\tan^{-1}\left(\frac{\sin x}{e^{px}+\cos x}\right) &= \frac{\pi^2}{12(p^2+1)};\\ \int dx\log(1+2e^{-px}\cos x+e^{-2px}) &= \frac{p\pi^2}{6(p^2+1)}\end{aligned}\right\}\ldots\ldots\ldots\ldots(22)$$

$$\left.\begin{aligned}\int dx\tan^{-1}\left(\frac{\sin px}{e^{x}+\cos px}\right) &= \frac{p\pi^2}{12(p^2+1)};\\ \int dx\log(1+2e^{-x}\cos px+e^{-2x}) &= \frac{\pi^2}{6(p^2+1)}\end{aligned}\right\}\ldots\ldots\ldots\ldots(23)$$

If we integrate $\int e^{-ax}\,dx\cos rx = \frac{a}{a^2+r^2}$ for r, we shall find $\int\frac{dx}{x}e^{-ax}\sin rx = \tan^{-1}\frac{r}{a}$. Make $a=r=1, 2, 3, etc.$; multiply by $h, h^2, h^3, etc.$, and sum; also multiply by $h, \frac{h^2}{2}, \frac{h^3}{3}, etc.$, and sum:

$$\int\frac{dx}{x}\,\frac{e^x\sin x}{e^{2x}-2he^x\cos x+h^2} = \frac{\pi}{4}\cdot\frac{1}{1-h}\ldots\ldots\ldots\ldots(24)$$

$$\int\frac{dx}{x}\tan^{-1}\left(\frac{h\sin x}{e^x-h\cos x}\right) = \frac{\pi}{4}\log\frac{1}{1-h}\ldots\ldots\ldots\ldots(25)$$

Make $a=r=1, 3, 5, etc.$; multiply by $h, -\frac{h^3}{3}, \frac{h^5}{5}, etc.$, and sum:

$$\int\frac{dx}{x}\log\left(\frac{e^{2x}+2he^x\sin x+h^2}{e^{2x}-2he^x\sin x+h^2}\right)=\pi\tan^{-1}h\ldots\ldots\ldots\ldots(26)$$

Multiply by $h, \frac{h^3}{3}, \frac{h^5}{5}$, and sum, a and r as in the last:

$$\int\frac{dx}{x}\tan^{-1}\left(\frac{2he^x\sin x}{e^{2x}-h^2}\right) = \frac{\pi}{4}\log\left(\frac{1+h}{1-h}\right)\ldots\ldots\ldots\ldots(27)$$

Others more general, as in (17), (18), *etc.*, might be found here.

Taking now the known integrals $\int x^a\,dxe^{-mx}\cos nx = \frac{P(a)}{r^{a+1}}\cos(a+1)\theta$, $\int x^a\,dxe^{-mx}\sin nx = \frac{P(a)}{r^{a+1}}\sin(a+1)\theta$, where $P(a)=1\cdot2\cdot3\ldots a$, $\tan\theta = \frac{n}{m} = p$ (suppose), $r=(m^2+n^2)^{\frac{1}{2}} = (1+p^2)^{\frac{1}{2}}\,m = \frac{m}{\cos\theta}$. Let $\Sigma\frac{1}{n^b} = \frac{1}{1^b}+\frac{1}{2^b}+\frac{1}{3^b}+\ldots$ to infinity, $\Sigma\frac{(-1)^{n+1}}{n^b} = \frac{1}{1^b}-\frac{1}{2^b}+\frac{1}{3^b}-\ldots$

These sums are known when b is an even integer.

First, let a be an odd integer, and make $m=1, 2, 3, etc.$ Summing by (i) and (k), we find

$$\int x^a\,dx\frac{e^x\cos px-1}{e^{2x}-2e^x\cos px+1} = P(a)\cos(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{1}{n^{a+1}}\ldots(28)$$

$$\int x^a\,dx\frac{e^x\sin px}{e^{2x}-2e^x\cos px+1} = P(a)\sin(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{1}{n^{a+1}}\ldots(29)$$

$$\int x^a\,dx\frac{e^x\cos px+1}{e^{2x}+2e^x\cos px+1}=\Gamma(a)\cos(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{(-1)^n}{n^{a+1}}\ldots(30)$$

$$\int x^a\,dx\frac{e^x\sin px}{e^{2x}+2e^x\cos px+1}=\mathrm{P}(a)\sin(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{(-1)^n}{n^{a+1}}\ldots(31)$$

Let a be even, m as before, multiply by $1, \pm\frac{1}{2}, \frac{1}{3}$, *etc.*, and sum by (c), (d), (g), (h):

$$\int x^a\,dx\log(1-2e^{-x}\cos px+e^{-2x})=-2\mathrm{P}(a)\cos(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{1}{n^{a+2}}\ldots(32)$$

$$\int x^a\,dx\log(1+2e^{-x}\cos px+e^{-2x})=2\mathrm{P}(a)\cos(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{(-1)^{n+1}}{n^{a+2}}\ldots(33)$$

$$\int x^a\,dx\tan^{-1}\left(\frac{\sin px}{e^x-\cos px}\right)=\mathrm{P}(a)\sin(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{1}{n^{a+2}}\ldots\ldots(34)$$

$$\int x^a\,dx\tan^{-1}\left(\frac{\sin px}{e^x+\cos px}\right)=\mathrm{P}(a)\sin(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{1}{n^{a+2}}\ldots\ldots(35)$$

Again, let a be even, $m=1, 3, 5$, *etc.*; multiply by $1, \pm\frac{1}{3}, \frac{1}{5}$, *etc.*; and sum by (a), (b), (e) (f).

$$\int x^a\,dx\log\left(\frac{e^{2x}+2e^x\cos px+1}{e^{2x}-2e^x\cos px+1}\right)=4\mathrm{P}(a)\cos(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{1}{(2n+1)^{a+2}}\ldots(36)$$

$$\int x^a\,dx\tan^{-1}\left(\frac{2e^x\sin px}{e^{2x}-1}\right)=2\mathrm{P}(a)\sin(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{1}{(2n+1)^{a+2}}\ldots(37)$$

$$\int x^a\,dx\tan^{-1}\left(\frac{2e^x\cos px}{e^{2x}-1}\right)=2\mathrm{P}(a)\cos(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{(-1)^{n+1}}{(2n+1)^{a+2}}\ldots(38)$$

$$\int x^a\,dx\log\left(\frac{e^{2x}+2e^x\sin px+1}{e^{2x}-2e^x\sin px+1}\right)=4\mathrm{P}(a)\sin(a+1)\theta(\cos\theta)^{a+1}\,\Sigma\frac{(-1)^{n+1}}{(2n+1)^{a+2}}\ldots(39)$$

In the last four $n=0, 1, 2$, *etc.*, and the sums denoted by Σ are known.

The accuracy of the preceding results will not be affected by a want of convergency in the series. But errors may arise from the first members of such series as (g) and (h) not representing the value of their second members through the whole extent of the integral. This would be the case with (4) for certain values of h.

The preceding list of integrals might be greatly extended, but I shall only indicate very briefly an easy method of finding indefinite integrals.

In $\int dx e^{-rx}=-\frac{1}{r}e^{-rx}$, make $r=r, 2r, 3r$, *etc*, and multiply the results by h, h^2, h^3, *etc.*, and sum; which gives

$$\int\frac{dx}{e^{rx}+h}=-\frac{1}{rh}\log(1+he^{-rx})\ldots\ldots\ldots\ldots(1')$$

In the last change h into $e^{\theta\sqrt{-1}}$ and $e^{-\theta\sqrt{-1}}$, and subtract results.

Also change h into $(-h)$, and repeat the operations:

$$\int\frac{dx}{e^{2rx}+2\cos\theta e^{rx}+1}=\frac{\cot\theta}{r}\tan^{-1}\left(\frac{\sin\theta e^{-rx}}{1+\cos\theta e^{-rx}}\right)$$
$$-\frac{1}{2r}\log(1+2\cos\theta e^{-rx}+e^{-2rx})\ldots\ldots(2')$$

$$\int\frac{dx}{e^{2rx}-2\cos\theta e^{rx}+1}=-\frac{\cot\theta}{r}\tan^{-1}\left(\frac{\sin\theta e^{-rx}}{1-\cos\theta e^{-rx}}\right)$$
$$-\frac{1}{2r}\log(1-2\cos\theta e^{-rx}+e^{-2rx})\ldots\ldots(3')$$

We may find $\int\frac{e^{rx}\,dx}{e^{2rx}+2\cos\theta e^{rx}+1}$ and $\int\frac{e^{rx}dx}{e^{2rx}-2\cos\theta e^{rx}+1}$ in a similar way.

If in (1′) we change r into $p+q\sqrt{-1}$ and $p-q\sqrt{-1}$, and add and subtract the results, we shall find the values of

$$\int\frac{dx(e^{px}\cos qx+h)}{e^{2px}+2he^{px}\cos qx+h^2}\text{ and }\int\frac{dxe^{px}\sin qx}{e^{2px}+2he^{px}\cos qx+h^2}.$$

In a similar way we may find many indefinite integrals, which might have been thought inexpressible in finite terms.

Gunthwaite Hall, near Penistone, June 20, 1844.

PROPERTIES OF THE PARABOLA.

[*Mr. Rutherford.*]

Let the curve be referred to the principal diameter, and a perpendicular to it through the vertex of the curve, as axes of co-ordinates; then $y^2=4mx$ is the equation of the curve.

Let $x_1y_1,\ x_2y_2,\ x_3y_3$ be the co-ordinates of any three points in the parabola;
$r_1,\ r_2,\ r_3$ the radii vectores at these points;
$\alpha,\ \beta,\ \gamma$ the angles made by the tangents at the three points with the axis of x;
$\mathrm{x}_1\mathrm{y}_1,\ \mathrm{x}_2\mathrm{y}_2,\ \mathrm{x}_3\mathrm{y}_3$ the co-ordinates of the points of intersection of the three tangents;
$p,\ q,\ r$ the co-ordinates of the centre, and radius of a circle passing through the three intersections of the tangents;
$f_1,\ f_2,\ f_3$ the focal distances of the three points of intersection of the tangents;
$\phi_1,\ \phi_2,\ \phi_3$ the angles formed by the lines $f_1f_2,\ f_1f_3,\ f_2f_3$ respectively;
$d_1,\ d_2,\ d_3$ the sides of the triangle formed by the three tangents;
$p_1,\ p_2,\ p_3$ the perpendiculars from the focus upon the tangents.

Then by the equation of the curve, and the known equation of the tangent at any point in the curve, we have

$$\left.\begin{aligned}y_1^2&=4m\,x_1\\y_2^2&=4m\,x_2\\y_3^2&=4m\,x_3\end{aligned}\right\}\ldots\ldots\ldots\ldots(1)\qquad\begin{aligned}yy_1&=2m(x+x_1)\ldots\ldots\ldots(2)\\yy_2&=2m(x+x_2)\ldots\ldots\ldots(3)\\yy_3&=2m(x+x_3)\ldots\ldots\ldots(4)\end{aligned}$$

And the co-ordinates of the points of intersection of (2,3); (2,4); (3,4)

$$\left.\begin{aligned} x_1 &= \frac{y_1 y_2}{4m} = \sqrt{x_1 x_2} \\ x_2 &= \frac{y_1 y_3}{4m} = \sqrt{x_1 x_3} \\ x_3 &= \frac{y_2 y_3}{4m} = \sqrt{x_2 x_3} \end{aligned}\right\}\ldots\ldots(5) \qquad \left.\begin{aligned} y_1 &= \frac{y_1+y_2}{2} = \sqrt{m}(\sqrt{x_1}+\sqrt{x_2}) \\ y_2 &= \frac{y_1+y_3}{2} = \sqrt{m}(\sqrt{x_1}+\sqrt{x_3}) \\ y_3 &= \frac{y_2+y_3}{2} = \sqrt{m}(\sqrt{x_2}+\sqrt{x_3}) \end{aligned}\right\}\ldots(6)$$

Now the equation of the circle which passes through the three points of intersection of the tangents is

$$x^2+y^2-2px-2qy+p^2+q^2-r^2=0\ldots\ldots\ldots\ldots\ldots(7)$$

and since this circle passes through the points x_1y_1, x_2y_2, x_3y_3, the equation (7) must be satisfied by the values of these co-ordinates as given in (5, 6); hence we have

$$y_1^2y_2^2+4m^2(y_1+y_2)^2-8mpy_1y_2-16m^2q(y_1+y_2)+16m^2(p^2+q^2-r^2)=0\ldots(8)$$

$$y_1^2y_3^2+4m^2(y_1+y_3)^2-8mpy_1y_3-16m^2q(y_1+y_3)+16m^2(p^2+q^2-r^2)=0\ldots(9)$$

$$y_2^2y_3^2+4m^2(y_2+y_3)^2-8mpy_2y_3-16m^2q(y_2+y_3)+16m^2(p^2+q^2-r^2)=0\ldots(10)$$

Subtract (9) and (10) severally from (8), and divide the results by (y_2-y_3) and (y_1-y_3) respectively, then there arise

$$y_1^2(y_2+y_3)+4m^2(2y_1+y_2+y_3)-8mpy_1-16m^2q=0\ \ldots\ldots\ldots(11)$$

$$y_2^2(y_1+y_3)+4m^2(y_1+2y_2+y_3)-8mpy_2-16m^2q=0\ \ldots\ldots\ldots(12)$$

Subtracting (12) from (11) gives at once, after simple reductions,

$$p=\frac{m}{2}+\frac{y_1y_2+y_2y_3+y_3y_1}{8m}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Hence also, $q=\dfrac{y_1+y_2+y_3}{4}-\dfrac{y_1y_2y_3}{16m^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$

Substitute these values of p and q partially in (8) and we readily obtain the equation

$$p^2+q^2-r^2=\frac{y_1y_2+y_2y_3+y_3y_1}{4}=2mp-m^2,\text{ by (13)}\ldots\ldots\ldots(15)$$

Or again from (15) we have the equation

$$r^2=(p-m)^2+q^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

which proves that *the circle circumscribing the triangle formed by the intersections of the three tangents passes through the focus of the parabola.* Since the circle must either cut the axis in two points, or touch it at the focus, we have, in the latter case, $p=m$, and $r=q$, and in the former the distance of the second point of intersection from the vertex is evidently

$$x=m+2(p-m)=2p-m=\frac{y_1y_2+y_2y_3+y_3y_1}{4m},\text{ by (13)};$$

$$\text{or, } x=\sqrt{x_1x_2}+\sqrt{x_2x_3}+\sqrt{x_3x_1}^{*},\text{ by equations (1)}\ldots\ldots\ldots\ldots(17)$$

* It must be understood throughout the investigation that if any point of contact of the tangent is on the negative side of the axis, the corresponding ordinate must be taken negative as well as the square root of the corresponding abscissa.

In (16) restore the values of p and q, and we have the equation

$$256m^4r^2=(4m^2+y^2_1)(4m^2+y^2_2)(4m^2+y^2_3)=64m^3(m+x_1)(m+x_2)(m+x_3);$$

$$\therefore\ 4mr^2=(m+x_1)(m+x_2)(m+x_3)=r_1r_2r_3 \ldots\ldots (18)$$

$$\text{or, } 2r=\left(\frac{r_1r_2r_3}{m}\right)^{\frac{1}{2}}.$$

Again, by (2, 3, 4) we get

$$\cot\alpha=\frac{y_1}{2m}\ \therefore\ \operatorname{cosec}^2\alpha=\frac{m+x_1}{m}\ \text{ or } m\operatorname{cosec}^2\alpha=m+x_1;$$

$$\cot\beta=\frac{y_2}{2m}\quad \operatorname{cosec}^2\beta=\frac{m+x_2}{m}\ \text{ or } m\operatorname{cosec}^2\beta=m+x_2;$$

$$\cot\gamma=\frac{y_3}{2m}\quad \operatorname{cosec}^2\gamma=\frac{m+x_3}{m}\ \text{ or } m\operatorname{cosec}^2\gamma=m+x_3;$$

hence, by (18) we get

$$4mr^2=m^3\operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma;$$

$$\therefore\ r=\frac{m}{2}\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma \ldots\ldots (19)$$

Having determined the values of p, q, r, the equation of the circle in (7) becomes either

$$8m^2(x^2+y^2)-2m(4m^2+y_1y_2+y_2y_3+y_1y_3)x-\{4m^2(y_1+y_2+y_3)-y_1y_2y_3\}y$$
$$+2m^2(y_1y_2+y_2y_3+y_1y_3)=0 \ldots (20)$$

$$\text{or, } x^2+y^2-(m+\sqrt{x_1x_2}+\sqrt{x_2x_3}+\sqrt{x_1x_3})x-m^{-\frac{1}{2}}\{m(\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3})$$
$$-\sqrt{x_1x_2x_3}\}y+m(\sqrt{x_1x_2}+\sqrt{x_2x_3}+\sqrt{x_1x_3})=0 \ldots (21)$$

In (20) and (21) make $y=0$, and we get either

$$4mx^2-(4m^2+y_1y_2+y_2y_3+y_1y_3)x+m(y_1y_2+y_2y_3+y_1y_3)=0,$$

$$\text{or } x^2-(m+\sqrt{x_1x_2}+\sqrt{x_2x_3}+\sqrt{x_1x_3})x+m(\sqrt{x_1x_2}+\sqrt{x_2x_3}+\sqrt{x_1x_3})=0,$$

which being written in a factorial form, give

$$\left.\begin{array}{l}(x-m)\{4mx-(y_1y_2+y_2y_3+y_1y_3)\}=0\\ \text{or}\quad (x-m)\{x-(\sqrt{x_1x_2}+\sqrt{x_2x_3}+\sqrt{x_1x_3})\}=0\end{array}\right\}\ldots\ldots (22)$$

Consequently we must have either

$$x-m=0,\quad \text{or } x-(\sqrt{x_1x_2}+\sqrt{x_2x_3}+\sqrt{x_1x_3})=0.$$

From the former of these we have $x=m$, and hence the circle passes through the focus of the parabola; from the other we get

$$x=\sqrt{x_1x_2}+\sqrt{x_2x_3}+\sqrt{x_1x_3} \ldots\ldots (23)$$

which determines the point where the circumscribing circle again cuts the axis of x. These results are very remarkable, and the latter, which is *independent* of the value of the parameter of the parabola, has not, as far as we know, been noticed by writers on this subject. We have already obtained the same results in (16) and (17), from different, though obvious, considerations.

If the circle cut also the axis of y, the distance of the points of intersection from the origin will be found from the equation (20) by making in it $x=0$. This gives

$$8m^2y^2-\{4m^2(y_1+y_2+y_3)-y_1y_2y_3\}y+2m^2(y_1y_2+y_2y_3+y_1y_3)=0 \ldots (24)$$

and the two roots (if real) of this equation are the distances sought.

When the circle only touches the axis of y, we must have the condition

$$64m^4\{y_1y_2+y_2y_3+y_1y_3\}=\{4m^2(y_1+y_2+y_3)-y_1y_2y_3\}^2 \text{........(25)}$$

If two of the points in the curve are fixed, whilst the position of the third is arbitrary, the locus of the *centre* of the circumscribing circle is a *determinate straight line.*

For if x_3y_3 be the arbitrary point, the elimination of y_3 from the equations (13) and (14) will furnish the equation of the line which is the locus of the centre of the circumscribing circle. From (13,14) we get

$$y_3(y_1+y_2)=8mp-(4m^2-y_1y_2)$$
$$y_3(4m^2-y_1y_2)=4m^2(4q-y_1-y_2),$$

and eliminating y_1, we obtain the equation

$$q=\frac{4m^2-y_1y_2}{2m(y_1+y_2)}p+\frac{y_1+y_2}{4}-\frac{16m^4-y_1^2y_2^2}{16m^2(y_1+y_2)} \text{........(26)}$$

which is that of a straight line given in position, and the locus of the centre of the circumscribing circle. This equation will be modified when the two given points have particular positions in the curve.

If the point x_2y_2 is at the extremity of the focal ordinate; then $y_2=2m$; and

$$q=\frac{2m-y_1}{2m+y_1}p+\frac{y_1}{2} \text{........................(27)}$$

If the point x_1y_1 is at the vertex; then $y_1=0$, and (26) gives

$$q=\frac{2m}{y_2}p+\frac{y_2^2-4m^2}{4y_2} \text{......................(28)}$$

And if one point is at the vertex, and the other at the extremity of the focal ordinate; $y_1=0$, and $y_2=2m$, and the equation reduces to

$$q=p \text{.........................(29)}$$

hence the locus of the centre, in this case, passes through the origin, and makes an angle of 45° with either axis.

The equation of the straight line which joins the focus and the centre of the circumscribing circle may also be readily determined.

The co-ordinates of the centre are p, q, and those of the focus are m, 0; hence the equation of the specified line is

$$y(m-p)+q(x-m)=0 \text{......................(30)}$$

But $p=\dfrac{m}{2}+\dfrac{y_1y_2+y_2y_3+y_1y_3}{8m}=\dfrac{m}{2}(1+\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\alpha\cot\gamma)$;

and $q=\dfrac{y_1+y_2+y_3}{4}-\dfrac{y_1y_2y_3}{16m^2}=\dfrac{m}{2}(\cot\alpha+\cot\beta+\cot\gamma-\cot\alpha\cot\beta\cot\gamma)$;

hence (30) becomes

$$y=\frac{q}{p-m}(x-m)=\frac{\cot\alpha+\cot\beta+\cot\gamma-\cot\alpha\cot\beta\cot\gamma}{\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\alpha\cot\gamma-1}(x-m);$$

$$\text{or } y=-\cot(\alpha+\beta+\gamma)(x-m) \text{..................(31)}$$

From this we deduce the remarkable fact, that when the *sum* of the angles which the tangents make with the axis of x is constant, the centres of all the circumscribing circles lie in the same straight line.

To determine the sides and area of the triangle formed by the intersections of the tangents, we have

$$d_1=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\frac{y_2-y_3}{4m}\sqrt{4m^2+y^2_1}=m\left(\frac{y_2}{2m}-\frac{y_3}{2m}\right)\left(1+\frac{y_1^2}{4m^2}\right)^{\frac{1}{2}}$$

$$\text{or } d_1=m\,(\cot\beta-\cot\gamma)\operatorname{cosec}\alpha=m\,\frac{\sin(\gamma-\beta)}{\sin\alpha\,\sin\beta\,\sin\gamma}\text{ ; and similarly}$$

$$d_2=m\frac{\sin(\gamma-\alpha)}{\sin\alpha\,\sin\beta\,\sin\gamma};\quad d_3=m.\frac{\sin(\beta-\alpha)}{\sin\alpha\,\sin\beta\,\sin\gamma}.$$

Hence the *perimeter* of the triangle is

$$\left.\begin{aligned}d_1+d_2+d_3&=m\frac{\sin(\beta-\alpha)+\sin(\gamma-\alpha)+\sin(\gamma-\beta)}{\sin\alpha\,\sin\beta\,\sin\gamma}\\&=2r\{\sin(\beta-\alpha)+\sin(\gamma-\alpha)+\sin(\gamma-\beta)\}\end{aligned}\right\};\quad\ldots\ldots\ldots(32)$$

and its *area* is

$$\frac{d_1d_2d_3}{4r}=\frac{(y_1-y_2)\,(y_2-y_3)\,(y_1-y_3)}{16m}=\frac{m^2}{2}.\frac{y_1-y_2}{2m}.\frac{y_2-y_3}{2m}.\frac{y_1-y_3}{2m}$$

$$=\frac{m^2}{2}\,(\cot\alpha-\cot\beta)\,(\cot\beta-\cot\gamma)\,(\cot\alpha-\cot\gamma)$$

$$=\frac{m^2}{2}.\frac{\sin(\beta-\alpha)\,\sin(\gamma-\alpha)\sin(\gamma-\beta)}{\sin^2\alpha\,\sin^2\beta\,\sin^2\gamma}.\ldots\ldots\ldots\ldots(33)$$

Lastly, if ρ be the radius of the inscribed circle; we have

$$\rho=\frac{2\Delta}{d_1+d_2+d_3}=2r\frac{\sin(\beta-\alpha)\,\sin(\gamma-\beta)\,\sin(\gamma-\alpha)}{\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin(\gamma-\alpha)}$$

$$=2r\frac{\sin(\beta-\alpha)\sin(\gamma-\beta)\,\sin\{\pi-(\gamma-\alpha)\}}{\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin\{\pi-(\gamma-\alpha)\}};$$

But since the sum of the angles $\beta-\alpha$, $\gamma-\beta$, and $\pi-(\gamma-\alpha)$ is equal to two right angles, therefore

$$\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin\{\pi-(\gamma-\alpha)\}=4\cos\frac{\beta-\alpha}{2}\cos\frac{\gamma-\beta}{2}\cos\frac{\pi-(\gamma-\alpha)}{2}$$

$$=4\cos\frac{\beta-\alpha}{2}\cos\frac{\gamma-\beta}{2}\sin\frac{\gamma-\alpha}{2};$$

hence the above value of ρ becomes

$$\rho=\frac{r}{2}.\frac{\sin(\beta-\alpha)\sin(\gamma-\beta)\sin(\gamma-\alpha)}{\cos\frac{1}{2}(\rho-\alpha)\,\cos\frac{1}{2}(\gamma-\beta)\sin\frac{1}{2}(\gamma-\alpha)}$$

$$\text{or } \rho=4r\sin\frac{\beta-\alpha}{2}\sin\frac{\gamma-\beta}{2}\cos\frac{\gamma-\alpha}{2}.\ldots\ldots\ldots\ldots(34)$$

To determine the focal distances of the points of intersection of the tangents, we get

$$\left.\begin{aligned}f^2_1&=(x_1-m)^2+y^2_1=(m+x_1)(m+x_2)=r_1r_2\\f^2_2&=(x_2-m)^2+y^2_2=(m+x_1)(m+x_3)=r_1r_3\\f^2_3&=(x_3-m)^2+y^2_3=(m+x_2)(m+x_3)=r_2r_3\end{aligned}\right\}\ldots\ldots\ldots(35)$$

or, in terms of the angles α,β,γ, these focal distances are expressed thus:

$$f_1=m\operatorname{cosec}\alpha\operatorname{cosec}\beta;\ f_2=m\operatorname{cosec}\alpha\operatorname{cosec}\gamma;\ f_3=m\operatorname{cosec}\beta\operatorname{cosec}\gamma;$$

hence, taking the product of the equations in (31), we obtain the remarkable results;

$$f_1 f_2 f_3 = r_1 r_2 r_3 \dots\dots\dots\dots\dots\dots (36)$$

$$\text{or } f_1 f_2 f_3 = 4m r^2, \text{ by (18)} \dots\dots\dots\dots (37)$$

The lengths of the perpendiculars from the focus upon the tangents are

$$\left.\begin{aligned} p_1 &= \sqrt{m(m+x_1)} = \sqrt{mr_1} = m \operatorname{cosec}\alpha \\ p_2 &= \sqrt{m(m+x_2)} = \sqrt{mr_2} = m \operatorname{cosec}\beta \\ p_3 &= \sqrt{m(m+x_3)} = \sqrt{mr_3} = m \operatorname{cosec}\gamma \end{aligned}\right\} \dots\dots\dots\dots (38)$$

Hence we have

$$p_1 p_2 p_3 = m^{\frac{3}{2}} \sqrt{r_1 r_2 r_3} = m^3 \operatorname{cosec}\alpha \operatorname{cosec}\beta \operatorname{cosec}\gamma,$$

$$\text{or } \quad p_1 p_2 p_3 = 2m^2 r, \text{ by (19)} \dots\dots\dots\dots\dots\dots\dots\dots (39)$$

The equations of the lines r_1, f_1, and r_2 are respectively

$$y = \frac{y_1}{x_1 - m}(x-m); \quad y = \frac{y_1}{x_1 - m}(x-m); \quad \text{and } y = \frac{y_2}{x_2 - m}(x-m);$$

and if $\theta_1, \theta_2, \theta_3$, denote the angles of the triangle, formed by the tangent at the point $x_1 y_1$, and the lines r_1, f_1, opposite these sides respectively; and $\omega_1, \omega_2, \omega_3$ denote the angles of the triangle, formed by the tangent at the point $x_2 y_2$, and the lines f_1 and r_2, opposite to the tangent, f_1, r_2 respectively; then from the above equations, and those of the tangents at $x_1 y_1$, $x_2 y_2$ given in (2,3) we have

$$\tan\theta_3 = \left(\frac{y_1}{x_1 - m} - \frac{2m}{y_1}\right) \div \left(1 + \frac{2m}{x_1 - m}\right) = \frac{2m}{y_1};$$

$$\tan\omega_3 = \left(\frac{y_1}{x_1 - m} - \frac{2m}{y_2}\right) + \left(1 + \frac{2my_1}{y_2(x_1 - m)}\right) = \frac{2m}{y_1} = \tan\theta_3;$$

hence $\theta_3 = \omega_3$; and in a similar manner it is shown that $\theta_1 = \omega_1$ and $\theta_2 = \omega_2$; consequently the triangles of which the angles are $\theta_1, \theta_2, \theta_3$ and $\omega_1, \omega_2, \omega_3$ are equiangular, and it has been shown in (35) that $f_1^2 = r_1 r_2$; hence, and from equations (36), (39), and (37) we deduce the following theorems:—

(1). *If straight lines be drawn from the focus to the points of contact and intersection of any two tangents to a parabola, the two triangles thus formed are equiangular, and the distance of the point of intersection of the tangents from the focus is a mean proportional between the radii vectores to the points of contact of the tangents.*

(2). *The continued product of the lines drawn from the focus to the three points of intersection of any three tangents to a parabola, is equal to the continued product of the lines drawn from the focus to the three points of contact of the tangents.*

(3). *The continued product of the perpendiculars from the focus upon the three tangents is equal to the product of the square of the distance of the focus from the vertex, and the diameter of the circle passing through the three points of intersection of the tangents.*

(4). *The continued product of the lines drawn from the focus to the points of intersection of any three tangents, is equal to the product of the parameter and the square of the radius of the circle passing through the three points of intersection of the tangents.*

By taking any number of points in the curve, several of the foregoing properties may be generalized, but we shall only add the subsequent investigation of the equation of the circle passing through any three points in the curve. This investigation, and the equations marked (19), (31) and (32) in the preceding pages, are due to *Mr. James Reid, of Prior's Salford*.

Let the equation of the circle passing through the points x_1y_1, x_2y_2, x_3y_3, be

$$x^2+y^2+Ax+By+C=0 \dots\dots\dots\dots (a)$$

$$\therefore\quad x^2_1+y^2_1+Ax_1+By_1+C=0 \dots\dots\dots\dots (b)$$

$$x^2_2+y^2_2+Ax_2+By_2+C=0 \dots\dots\dots\dots (c)$$

$$x^2_3+y^2_3+Ax_3+By_3+C=0 \dots\dots\dots\dots (d)$$

From (b) and (c) we get

$$x^2_1-x^2_2+y^2_1-y^2_2+A(x_1-x_2)+B(y_1-y_2)=0,$$

$$\text{or}\quad \frac{y^4_1-y^4_2}{16m^2}+y^2_1-y^2_2+A\frac{y^2_1-y^2_2}{4m}+B(y_1-y_2)=0,$$

$$\therefore\quad \frac{y^2_1+y^2_2}{16m^2}+1+\frac{A}{4m}+\frac{B}{y_1+y_2}=0 \dots\dots\dots (e)$$

Similarly from (b) and (d),

$$\frac{y^2_1+y^2_3}{16m^2}+1+\frac{A}{4m}+\frac{B}{y_1+y_3}=0 \dots\dots\dots (f)$$

Subtracting (f) from (e), and reducing, we get at once

$$B=\frac{(y_1+y_2)(y_1+y_3)(y_2+y_3)}{16m^2}.$$

$$\text{From eq.}(f),\ \frac{A}{4m}=-\frac{y^2_1+y^2_3+16m^2}{16m^2}-\frac{(y_1+y_2)(y_2+y_3)}{16m^2};$$

$$\text{hence,}\ A=-\frac{y^2_1+y^2_2+y^2_3+y_1y_2+y_1y_3+y_2y_3+16m^2}{4m}.$$

From (b), $C=-\dfrac{y_1y_2y_3(y_1+y_2+y_3)}{16m^2}$; and consequently (a) becomes

$$x^2+y^2-\frac{y^2_1+y^2_2+y^2_3+y_1y_2+y_1y_3+y_2y_3+16m^2}{4m}x+\frac{(y_1+y_2)(y_1+y_3)(y_2+y_3)}{16m^2}y$$
$$-\frac{y_1y_2y_3(y_1+y_2+y_3)}{16m^2}=0 \dots (g)$$

which is the equation of the circle passing through the points x_1y_1, x_2y_2, and x_3y_3. Hence we deduce

(1). *If in a parabola any three points be taken so that the sum of their ordinates is zero, the circle described through these points will pass through the vertex.*

(2). *If a parabola be placed horizontally, and three equal heavy particles be placed on the curve so as to produce equilibrium round*

the axis, the three particles lie in the circumference of a circle that passes through the vertex.

Hence we may deduce, also, the *equation of the osculating circle* at any point in the parabola. For if all the points coincide, then $y_1=y_2=y_3$ and the equation (g) reduces to either

$$\left.\begin{array}{l} x^2+y^2-\dfrac{8m^2+3y_1^2}{2m}x+\dfrac{y_1^3}{2m^2}y-\dfrac{3y_1^4}{16m^2}=0 \\ \text{or, } x^2+y^2-2(2m+3x_1)x+4x_1\left(\dfrac{x_1}{m}\right)^{\frac{1}{2}}y-3x_1^2=0 \end{array}\right\} \dots\dots\dots (h).$$

And if α, β are the co-ordinates of the centre, and ρ the radius of the osculating circle; then

$$\alpha=\frac{8m^2+3y_1^2}{4m}=2m+3x_1=3(m+x_1)-m=3r_1-m$$

$$\beta=-\frac{y_1^3}{4m^2}=-\frac{x_1y_1}{m}=-2m^{-\frac{1}{2}}x_1^{\frac{3}{2}}$$

$$\rho^2=\alpha^2+\beta^2+\frac{3y_1^4}{16m^2}=\frac{(8m^2+3y_1^2)^2}{16m^2}+\frac{x_1^2y_1^2}{m^2}+\frac{3y_1^4}{16m^2}$$

$$=\frac{64m^6+48m^4y_1^2+12m^2y_1^4+y_1^6}{16m^4}=\frac{(4m^2+y_1^2)^3}{16m^4}.$$

Hence, $$\rho=\pm\frac{(4m^2+y_1^2)^{\frac{3}{2}}}{4m^2}=\pm2m^{-\frac{1}{2}}(m+x_1)^{\frac{3}{2}}=\pm2\left(\frac{r_1^3}{m}\right)^{\frac{1}{2}}=\pm2r_1\sqrt{\frac{r_1}{m}},$$

which is an expression for the radius of curvature at any point in the parabola, r_1 being the distance of the proposed point from the focus.

If y_1 or $x_1=0$, or if $r_1=m$; then $\rho=2m$, the radius of curvature at the vertex; and if $r_1=2m$, then $\rho=4m\sqrt{2}$, the radius of curvature at the extremity of the focal ordinate.

MATHEMATICAL EXERCISES—*(continued.)*

25.—*Mr. Matthew Collins, Limerick.*

Given the bisectors of the interior and exterior angles at the vertex, and the sum or difference of the sides, to construct the plane triangle by elementary geometry.

26.—*Mr. G. W. Hearn, Royal Military College, Sandhurst.*

If $a, b, c, \dots\dots k$ be any n quantities; then will

$$\frac{abc..}{a(a+b)(a+b+c)..}+\frac{bac..}{b(b+a)(b+a+c)..}+\frac{cab..}{c(c+a)(c+a+b)..}+etc.=1.$$

Required an elementary algebraical investigation.

27.—*C. F. B.*

Into a cubical cistern, eight feet deep, and having an unknown leak, water is poured from two pumps, worked by two men A and B. They pump together till the vessel is half filled, when B falls asleep. A continues pumping till it is $\frac{3}{4}$ filled, and then goes away. B afterwards waking finds the cistern still half full, and after pumping till it is again $\frac{3}{4}$ filled, departs

also, and meeting with A charges him with leaving his work unfinished. They return together, and find the water $1\frac{1}{2}$ inches lower than when B left. The leak is now discovered and stopped: and by their joint efforts the vessel is filled in half the time they had worked together at first. They remark also that $10\frac{1}{2}$ hours had elapsed since they first began pumping, and that B had worked alone twice as long as A had. Supposing that a cubic foot contains $15\frac{5}{8}$ gallons, required the quantity of water thrown in by each pump, as well as the quantity discharged at the leak, *while one or both were pumping.*

28.—*Mr. Thomas Dobson, Totteridge, Herts.*

From a point P in the plane of a given quadrilateral figure ABCD, right lines, PQ, PR, PS, PT, are drawn parallel to lines given in position, to meet AB, BC, CD, AD, respectively. Determine the locus of P, when PR. PT : PS. PQ in a constant ratio.

29.—*Mr. Fenwick.*

Prove that the point $(z_1 y_1 x_1)$, and the plane

$$c\,z_1\,z+b\,y_1 y+(a\,x_1+e)\,x+e\,x_1=0,$$

have the relation of pole and polar, in respect of the surface

$$c\,z^2+b\,y^2+a\,x^2+2e\,x=0;$$

and thence deduce the equation of a tangent plane to the same surface.

30.—*Mr. Rutherford.*

Let ABC be a plane triangle and P any point in its plane; then if a, b, c denote the sides of the triangle respectively opposite to A, B, C; α, β, γ the respective distances of P from the same angular points, and Δ the area of a triangle whose three sides are α sinA, β sinB, γ sinC; it is required to prove that

$$2a^2 \operatorname{cosec}A \sin B \sin C = \alpha^2 \sin 2A+\beta^2 \sin 2B+\gamma^2 \sin 2C \pm 8\Delta;$$

the sign $+$ applying when the point P is within, and the sign $-$ when the point is without the triangle.

31.—*Mr. Philip Beecroft, Hyde—(reproposed.)*

Let O, O_1, O_2, O_3, be the centres of four circles in mutual contact on a plane; P, P_1, P_2, P_3, the centres of the four other circles touching each other mutually in the same points as the other four; O_4 the centre of another circle touching the circles centres O_1, O_2, O_3, and P_4 the centre of another touching the circles centres P_1, P_2, P_3. Let the circle centre O touch the circles centres O_1, O_2, O_3 in the points D, E, F, respectively, and the circle centre O_4 touch them in the points D_4, E_4, F_4 respectively; also let the circle centre P touch the circles centres P_1, P_2, P_3 in the points d, e, f, respectively, and the circle centre P_4 touch the same three in the points d_4, e_4, f_4, respectively. Then a circle will pass through the four points D, D_4, d, d_4; another will pass through the four points E, E_4, e, e_4; and another will pass through the four points F, F_4, f, f_4: also these three circles will intersect in two points U, V, and the six points O_4, P, O, P_4, U, V will be in a right line.

32.—*By Pen-and-Ink.*

Two straight lines being given in space, and a point in one of them: it is required to draw through the given point a line to meet the other given line, and make equal angles with them both.

SOLUTIONS OF MATHEMATICAL EXERCISES.

XVI.—*James Lockhart, Esq.*

Find the relation between the roots of the two equations

$$x^3 - bx = c\ldots.(1) \qquad x^3 + bx^2 = c^2\ldots.(2)$$

[First Solution.—*Mr. Hugh Godfray, Jersey.*]

Let α,β,γ be the three roots of (1); then by the theory of equations,

$$\alpha+\beta+\gamma=0;\ \alpha\beta+\alpha\gamma+\beta\gamma=-b,\ \alpha\beta\gamma=c;$$

whence,

$$-b=\ldots\ldots\ldots\ldots=\alpha\beta+\alpha\gamma+\beta\gamma,$$

$$0=(\alpha+\beta+\gamma)\alpha\beta\gamma=\alpha\beta.\alpha\gamma+\alpha\beta.\beta\gamma+\alpha\gamma.\beta\gamma,$$

$$c^2=\quad \alpha^2\beta^2\gamma^2\quad =\alpha\beta.\alpha\gamma.\beta\gamma.$$

But the equation whose roots are $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$ is evidently

$$x^3-(\alpha\beta+\alpha\gamma+\beta\gamma)\,x^2+(\alpha\beta.\alpha\gamma+\alpha\beta.\beta\gamma+\alpha\gamma.\beta\gamma)x-\alpha\beta.\alpha\gamma.\beta\gamma=0$$

$$\text{or, } x^3-(-b)x^2+0.x-c^2=0, \text{ or } x^3+bx^2=c^2;$$

which being identical with (2) shews that the roots of (2) are the products, two and two, of the roots of (1). In a similar manner it is proved that the roots of equation (1) are the products, two and two, of the roots of (2).

[Second Solution.—*Mr. Weddle, and similarly by Mr. S. Bills, Hawton.*]

The roots of these two equations are so related to each other, that if $\frac{c}{x}$ be written for x in either equation, the other equation will result; hence if r_1,r_2,r_3 be the roots of equation (1); then $\frac{c}{r_1}, \frac{c}{r_2}, \frac{c}{r_3}$ are the roots of (2), and *vice versa*. But by the theory of equations, $c=r_1r_2r_3$; hence the roots of *eq.* (2) are r_1r_2, r_1r_3, r_2r_3, that is, the roots of either equation are the products, two and two, of the roots of the other.

Good solutions were received from Messrs. Thomas Dobson, Totteridge; St. Andrew St. John, R. M. Academy; Matthew Collins, Limerick; and the proposer.

XVII.—*Mr. Rutherford.*

Eliminate θ from the equations

$$m\tan 2\theta - n\tan 2\phi = 0\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$m\ \cot\phi - n\ \cot\theta\ = p\,\mathrm{cosec}^2\theta\ldots\ldots\ldots\ldots(2)$$

and give the resulting equation in terms of ϕ, when

$$m = \mathrm{W}'b + \mathrm{W}a,\quad n = \mathrm{W}'b - \mathrm{W}a,\quad p = (\mathrm{W} + \mathrm{W}')r.$$

[Solution.—*Mr. John Laws, Newcastle-on-Tyne.*]

From equation (1) we have $n\cot 2\theta = m\cot 2\phi$, or, by the formula for the cotangent of a double arc,

$$n\cot\theta - n\tan\theta = m\cot\phi - m\tan\phi;$$

but by (2), $\quad p\,\text{cosec}^2\theta + n\cot\theta = m\cot\phi$
and substituting, $\quad p\,\text{cosec}^2\theta + n\tan\theta = m\tan\phi$ $\Big\}$ .... (3)

These equations (3) might be employed to eliminate θ; but this will be better accomplished in the following manner:

From (1) we get

$$n\tan 2\phi = m\tan 2\theta = \frac{2m\cot\theta}{\cot^2\theta - 1},$$

$$\therefore n\tan 2\phi\cot^2\theta - 2m\cot\theta = n\tan 2\phi \ldots\ldots\ldots\ldots\ldots (3)$$

And from (2), $\quad p\cot^2\theta + n\cot\theta = m\cot\phi - p \ldots\ldots\ldots\ldots (4)$

Eliminate $\cot^2\theta$ from these equations, and we obtain

$$\cot\theta = \frac{n(m\cot\theta - 2p)}{n^2 + 2mp\cot 2\phi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Substituting this value of $\cot\theta$ in (4), and dividing by p, gives

$$\cot^2\theta = \frac{n^2 + 2m(m\cot\phi - p)\cot 2\phi}{n^2 + 2mp\cot 2\phi} \ldots\ldots\ldots\ldots\ldots (6)$$

Equating the values of $\cot^2\theta$ from (5) and (6) we get

$$\begin{aligned} n^2(m\cot\phi - 2p)^2 &= (n^2 + 2mp\cot 2\phi)(n^2 - 2mp\cot 2\phi + 2m^2\cot\phi\cot 2\phi) \\ &= n^4 - 4m^2p^2\cot^2 2\phi + 2m^2\cot\phi\cot 2\phi(n^2 + 2mp\cot 2\phi) \\ &= n^4 - 4m^2p^2\cot^2 2\phi + m^2n^2(\cot^2\theta - 1) + 4m^3p\cot\phi\cot^2 2\phi. \end{aligned}$$

Squaring the member on the left, and cancelling, we obtain

$$\frac{n^2(m^2 - n^2 + 4p^2)}{4mp} = n^2\cot\phi + m(m\cot\phi - p)\cot^2 2\phi \ldots\ldots\ldots (7)$$

Writing for $\cot^2 2\phi$ its value in terms of $\cot\theta$, and arranging the result, gives

$$m^3p\cot^5\phi - m^2p^2\cot^4\phi - 2mp(m^2 - 2n^2)\cot^3\phi + \{(m^2 - n^2)(2p^2 - n^2) - 2n^2p^2\}\cot^2\phi$$
$$+ m^3p\cot\phi - m^2p^2 = 0 \ldots (8)$$

By writing for m, n, p their specified values, the final equation would be obtained in terms of $\cot\phi$, considered as the quantity to be determined. But if r be considered the unknown; then substituting in (7), and putting

$$n^2 + m^2\cot^2 2\phi = \text{K}^2,$$

we have $r^2(\text{W}+\text{W}')^2\text{K}^2 - r(\text{W}+\text{W}')(\text{W}'b + \text{W}a)\text{K}^2\cot\phi$
$+ \text{WW}'ab(\text{W}'b - \text{W}a)^2 = 0;$

$$\text{or,}\quad r^2 - \frac{(\text{W}'b + \text{W}a)\cot\phi}{\text{W}' + \text{W}}r + \frac{\text{WW}'ab(\text{W}'b - \text{W}a)^2}{(\text{W}+\text{W}')^2\text{K}^2} = 0,$$

an equation from which, when ϕ is known, r may be determined.

Mr. Thomas Dobson favoured us with a solution to this question.

XVIII.—*Mr. St. Andrew St. John, Gent. Cadet, R. M. Academy.*

Trace the curve whose equation is

$$y^2 = \frac{x^3 - b^3}{x - a}.$$

[SOLUTION.—*By Theta, and similarly by Mr. Thomas Dobson.*]

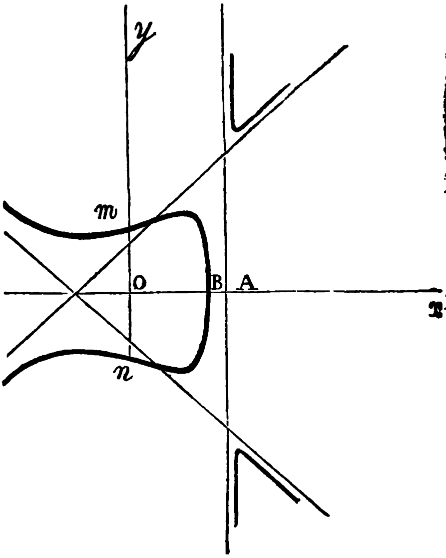
Take Ox and Oy as axes of co-ordinates; and since

$$y=\pm\left(\frac{x^3-b^3}{x-a}\right)^{\frac{1}{2}},$$

the curve is divided symmetrically by the axis of x.

Let $a > b$.

Then, when $x=0$, $y=\pm b\left(\frac{b}{a}\right)^{\frac{1}{2}}$; when $x<b$ (x being positive) y is real; and when $x=b$, $y=0$: hence, taking OB$=b$, and Om, On, each equal to $b\left(\frac{b}{a}\right)^{\frac{1}{2}}$, the curve extends from B to m, and from B to n.



Again, x being positive, when $x>b<a$, y is imaginary, and therefore no part of the curve lies between the ordinates at these points; when $x=a, y$ is infinite, and the ordinate at this point is therefore an asymptote to the curve: also when $x>a$, y has always two real values with opposite signs; two branches of the curve therefore extend indefinitely in this direction.

For *negative* values of x, the equation of the curve becomes

$$y=\pm\left(\frac{x^3+b^3}{x+a}\right)^{\frac{1}{2}};$$

hence for every value of x there will be two equal values with opposite signs for y, and when x is infinite y is infinite: it follows, then, that in this direction, also, there are two indefinite branches.

The equation of the curve when developed assumes the form

$$y=\pm x\left(1-\frac{a}{x}\right)^{-\frac{1}{2}}\left(1-\frac{b^3}{x^3}\right)^{\frac{1}{2}}$$

$$=\pm\left(x+\frac{a}{2}+\mathrm{A}x^{-1}+\mathrm{B}x^{-2}+\ldots\right);$$

rejecting, therefore, *negative powers* for x, we get

$$y=\pm\left(x+\tfrac{1}{2}a\right),$$

for the equation of two rectilinear asymptotes passing through $x=-\frac{1}{2}a$, and cutting the axes at angles $+45^\circ$, and -45°.

Differentiating,

$$\frac{dy}{dx}=\pm\frac{2x^3-3ax^2+b^3}{2(x-a)^{\frac{3}{2}}(x^3-b^3)^{\frac{1}{2}}};$$

whence at B, where $x=b$, we have tan X $=\infty$, or X $=90^\circ$; and therefore the curve cuts the axis of x at right angles at B. Also, when $x=0$; $\frac{dx}{dy}=\pm 2\left(\frac{a^3}{b^3}\right)^{\frac{1}{2}}$; consequently at m and n the curve cuts the axis of y at

angles $\tan^{-1}2\left(\frac{a^3}{b^3}\right)^{\frac{1}{3}}$ and $\tan^{-1}-2\left(\frac{a^3}{b^3}\right)^{\frac{1}{3}}$.

Again, the condition $\frac{dy}{dx}=0$, gives $2x^3-3ax^2+b^3=0$; hence, for $x=0$, $\frac{dy}{dx}=0$, gives b^3 as the result, which is *positive*; and for $x=b$, $\frac{dy}{dx}=0$, gives $3b^2(b-a)$, which is *negative*, since $a>b$; consequently one value of x in $\frac{dy}{dx}=0$, is between 0 and b, corresponding to a maximum or minimum ordinate (according as the second differential coefficient belonging to this point is negative or positive), between the points O and B.

Again, for $x=a$, $\frac{dy}{dx}=0$, gives b^3-a^3, which is negative; and for $x=2a$, it gives $4a^3+b^3$, which is positive; another maximum or minimum ordinate is therefore between $x=a$, and $x=2a$. Lastly, for $x=-b$, $\frac{dy}{dx}=0$, gives $-b^2(b+3a)$, which is negative, and since for $x=0$, $\frac{dy}{dx}=0$, gives a positive result, as has been shewn, the third maximum or minimum ordinate is between $x=0$, and $x=-b$.

Differentiating again with respect to x,

$$\frac{d^2y}{dx^2}=\pm\frac{3(a^2x^4-4b^3x^3+6ab^3x^2-4a^2b^3x+b^6)}{4(x-a)^{\frac{5}{3}}(x^3-b^3)^{\frac{3}{2}}};$$

in order, then, to find the *points of inflection*, we must put $\frac{d^2y}{dx^2}=0$, or $=\infty$, and reason with this equation as with $\frac{dy}{dx}=0$, for finding the maximum and minimum ordinates.*

THETA also enters into a discussion of the curve when $a<b$, but this we leave for the student's private exercise.

The proposer also favored us with an analogous discussion.

XIX.—*Mr. Thomas Weddle, Newcastle.*

Required the magnitude and position of the circles touching two sides of a triangle and the circumscribing circle.

[FIRST SOLUTION.— *Mr. Weddle, the proposer.*]

Def. A circle is said to touch the side of a triangle *internally* when it touches it (produced if necessary) on the same side as the inscribed circle, and *externally* when it touches on the other side.

Let ABC be a triangle (the student will readily sketch the figure) about which a circle (centre a) is described; twelve circles (radii $\rho_1, \rho'_1, \rho''_1, \rho'''_1, \rho_2, \rho'_2, \rho''_2, \rho'''_2, \rho_3, \rho'_3, \rho''_3, \rho'''_3$, and centres $R_1, R'_1, R''_1, R'''_1, R_2, R'_2, R''_2, R'''_2, R_3, R'_3, R''_3, R'''_3$) may be drawn to touch the circumscribing circle and two sides of the triangle. Of these let ρ_1, ρ_2, ρ_3 touch the circumscribing circle

* In connexion with the tracing of curves, we would recommend to the student's notice, Hind's valuable treatise on the Differential Calculus.

internally, and AB and AC, AB and BC, and AC and BC, respectively (internally.) The other circles will touch the circumscribing circle externally: let ρ'_1, ρ''_2, ρ'''_3 touch AB and AC, AB and BC, and AC and BC internally; ρ''_1, AC externally and AB internally; ρ'''_1, AB externally and AC internally, *etc.* (Here instead of the circle whose centre is R and radius ρ, I say, for brevity, the circle ρ.)

If O, O_1, O_2, O_3 be the centres of the circles of contact, it is obvious that A, O, R_1, O_1, R'_1, are in a straight line, as also R''_1, O_2, A, O_3, R'''_1. Draw QK and the radii R_1M, R'_1M', R''_1M'', R'''_1M''' perpendicular to AC. Then we shall have $AK = R.\sin B$, $QK = R\cos B$, $AM = \rho_1 \cot\frac{1}{2}A$, $AM' = \rho'_1 \cot\frac{1}{2}A$, $AM'' = \rho''_1 \tan\frac{1}{2}A$, and $AM''' = \rho_1''' \tan\frac{1}{2}A$. Moreover it is evident that

$$\begin{aligned} R_1Q^2 &= MK^2 + (R_1M - QK)^2 \\ R'_1Q^2 &= M'K^2 + (R'_1M' - QK)^2 \\ R''_1Q^2 &= M''K^2 + (R''_1M'' - QK)^2 \\ \text{and}\quad R'''_1Q^2 &= M''K^2 + (R'''_1M''' - QK)^2 \end{aligned}$$

$$\begin{aligned} \text{or,}\quad (R - \rho_1)^2 &= (\rho_1 \cot\tfrac{1}{2}A - R\sin B)^2 + (\rho_1 - R\cos B)^2 \\ (R + \rho'_1)^2 &= (\rho'_1 \cot\tfrac{1}{2}A - R\sin B)^2 + (\rho'_1 - R\cos B)^2 \\ (R + \rho''_1)^2 &= (\rho''_1 \tan\tfrac{1}{2}A - R\sin B)^2 + (\rho''_1 + R\cos B)^2 \\ (R + \rho'''_1)^2 &= (\rho'''_1 \tan\tfrac{1}{2}A + R\sin B)^2 + (\rho'''_1 - R\cos B)^2 \end{aligned}$$

Taking the first of these we have, after a little obvious reduction,

$$\frac{\rho_1\cot^2\frac{1}{2}A}{D} = \frac{\cos\frac{1}{2}A\sin B + \sin\frac{1}{2}A\cos B - \sin\frac{1}{2}A}{\sin\frac{1}{2}A} = \frac{\sin(\frac{1}{2}A + B) - \sin\frac{1}{2}A}{\sin\frac{1}{2}A}$$

$$= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}B}{\sin\frac{1}{2}A} = \frac{2\sin\frac{1}{2}B\cdot\sin\frac{1}{2}C}{\sin\frac{1}{2}A}$$

$$\therefore \rho_1 = \frac{2D\cdot\sin\frac{1}{2}B\cdot\sin\frac{1}{2}C}{\cos\frac{1}{2}A\cdot\cot\frac{1}{2}A}.$$

In a similar manner the others may be reduced, and tabulating the whole, we have

$$\rho_1 = \frac{2D\sin\frac{1}{2}B.\sin\frac{1}{2}C}{\cos\frac{1}{2}A.\cot\frac{1}{2}A},\quad \rho_2 = \frac{2D.\sin\frac{1}{2}A.\sin\frac{1}{2}C}{\cos\frac{1}{2}B.\cot\frac{1}{2}B},\quad \rho_3 = \frac{2D\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{1}{2}C.\cot\frac{1}{2}C},$$

$$\rho'_1 = \frac{2D\cos\frac{1}{2}B.\cos\frac{1}{2}C}{\cos\frac{1}{2}A.\cot\frac{1}{2}A},\quad \rho'_2 = \frac{2D\sin\frac{1}{2}A.\cos\frac{1}{2}C}{\sin\frac{1}{2}A.\tan\frac{1}{2}B},\quad \rho'_3 = \frac{2D\sin\frac{1}{2}A.\cos\frac{1}{2}B}{\sin\frac{1}{2}C.\tan\frac{1}{2}C},$$

$$\rho''_1 = \frac{2D\sin\frac{1}{2}B.\cos\frac{1}{2}C}{\sin\frac{1}{2}A.\tan\frac{1}{2}A},\quad \rho''_2 = \frac{2D\cos\frac{1}{2}A.\cos\frac{1}{2}C}{\cos\frac{1}{2}B.\cot\frac{1}{2}B},\quad \rho''_3 = \frac{2D\cos\frac{1}{2}A.\sin\frac{1}{2}B}{\sin\frac{1}{2}C.\tan\frac{1}{2}C},$$

$$\rho'''_1 = \frac{2D\cos\frac{1}{2}B.\sin\frac{1}{2}C}{\sin\frac{1}{2}A.\tan\frac{1}{2}A},\quad \rho'''_2 = \frac{2D\cos\frac{1}{2}A\sin\frac{1}{2}C}{\sin\frac{1}{2}B.\tan\frac{1}{2}B},\quad \rho'''_3 = \frac{2D\cos\frac{1}{2}A.\cos\frac{1}{2}B}{\cos\frac{1}{2}C.\cot\frac{1}{2}A},$$

......(1)*

* The following values of the twelve radii which are easily deduced from those given by Mr. Weddle (the notation being slightly changed), may be interesting to the student, viz:—

$\rho_1 = (s-a)\tan\frac{1}{2}A\sec^2\frac{1}{2}A$; $\rho'_1 = s\tan\frac{1}{2}A\sec^2\frac{1}{2}A$;
$\rho''_1 = (s-c)\cot\frac{1}{2}A\operatorname{cosec}^2\frac{1}{2}A$; $\rho'''_1 = (s-b)\cot\frac{1}{2}A\operatorname{cosec}^2\frac{1}{2}A$;
$\rho_2 = (s-b)\tan\frac{1}{2}B\sec^2\frac{1}{2}B$; $\rho'_2 = s\tan\frac{1}{2}B\sec^2\frac{1}{2}B$;
$\rho''_2 = (s-c)\cot\frac{1}{2}B\operatorname{cosec}^2\frac{1}{2}B$; $\rho'''_2 = (s-a)\cot\frac{1}{2}B\operatorname{cosec}^2\frac{1}{2}B$;
$\rho_3 = (s-c)\tan\frac{1}{2}C\sec^2\frac{1}{2}C$; $\rho'_3 = s\tan\frac{1}{2}C\sec^2\frac{1}{2}C$;
$\rho''_3 = (s-b)\cot\frac{1}{2}C\operatorname{cosec}^2\frac{1}{2}C$; $\rho'''_3 = (s-a)\cot\frac{1}{2}C\operatorname{cosec}^2\frac{1}{2}C$.

The circles (ρ_1) and (ρ'_1) have their centres in the line which bisects the angle A, (ρ''_1) and (ρ'''_1) in the line which bisects the exterior of A; and so for the others.

$$\therefore \rho_1.\rho'_1\,\rho''_1.\rho'''_1 = \rho_2.\rho'_2.\rho''_2.\rho'''_2 = \rho_3.\rho'_3.\rho''_3.\rho'''_3 = s(s-a)(s-b)(s-c) = \Delta^2.$$

F.

$$\left.\begin{array}{l}
-2+\cos^2\tfrac{1}{2}A+\cos^2\tfrac{1}{2}B+\cos^2\tfrac{1}{2}C=2\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C=\dfrac{r}{D}\\
2-\cos^2\tfrac{1}{2}A-\sin^2\tfrac{1}{2}B-\sin^2\tfrac{1}{2}C=2\sin\tfrac{1}{2}A.\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C=\dfrac{r_1}{D}\\
2-\sin^2\tfrac{1}{2}A-\cos^2\tfrac{1}{2}B-\sin^2\tfrac{1}{2}C=2\cos\tfrac{1}{2}A.\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C=\dfrac{r_2}{D}\\
2-\sin^2\tfrac{1}{2}A-\sin^2\tfrac{1}{2}B-\cos^2\tfrac{1}{2}C=2\cos\tfrac{1}{2}A.\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C=\dfrac{r_3}{D}
\end{array}\right\}\ldots.(2)$$

It may easily be shown (from $A+B+C=\pi$) that the two first members of (2) are equal, and that the two last are so, will be shown in the Horæ Geom., in next year's diary.

By means of (2), (1) takes a very simple form; thus by (1),

$$\rho_1\cos^2\tfrac{1}{2}A=\rho_2\cos^2\tfrac{1}{2}B=\rho_3\cos^2\tfrac{1}{2}C=2D\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C=r, \text{ by (2).}$$

$$\left.\begin{array}{l}
\therefore r=\rho_1\cos^2\tfrac{1}{2}A=\rho_2\cos^2\tfrac{1}{2}B=\rho_3\cos^2\tfrac{1}{2}C\\
r_1=\rho_1'\cos^2\tfrac{1}{2}A=\rho'_2\sin^2\tfrac{1}{2}B=\rho'_3\sin^2\tfrac{1}{2}C\\
r_2=\rho''_1\sin^2\tfrac{1}{2}A=\rho''_2\cos^2\tfrac{1}{2}B=\rho''_3\sin^2\tfrac{1}{2}C\\
r_3=\rho'''_1\sin^2\tfrac{1}{2}A=\rho'''_2\sin^2\tfrac{1}{2}B=\rho'''_3\cos^2\tfrac{1}{2}C
\end{array}\right\}\ldots\ldots\ldots\ldots(3)$$

By (1),

$$\rho'_1-\rho_1=\frac{2D\{\cos\tfrac{1}{2}B.\cos\tfrac{1}{2}C-\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C\}}{\cos\tfrac{1}{2}A\cot\tfrac{1}{2}A}=\frac{2D\cos\tfrac{1}{2}(B+C)}{\cos\tfrac{1}{2}A.\cot\tfrac{1}{2}A}=\frac{2D\sin\tfrac{1}{2}A}{\cos\tfrac{1}{2}A\cot\tfrac{1}{2}A}$$

$=2D\tan^2\tfrac{1}{2}A$, $\therefore$ $2D=(\rho'_1-\rho_1)\cot^2\tfrac{1}{2}A$. Similarly, $2D=(\rho''_1+\rho'''_1)\tan^2\tfrac{1}{2}A$;

$$\left.\begin{array}{l}
\therefore 2D=(\rho'_1-\rho_1)\cot^2\tfrac{1}{2}A=(\rho''_1+\rho'''_1)\tan^2\tfrac{1}{2}A\\
\qquad=(\rho''_2-\rho_2)\cot^2\tfrac{1}{2}B=(\rho'_2+\rho'''_2)\tan^2\tfrac{1}{2}B\\
\qquad=(\rho'''_3-\rho_3)\cot^2\tfrac{1}{2}C=(\rho'_3+\rho''_3)\tan^2\tfrac{1}{2}C
\end{array}\right\}\ldots\ldots\ldots(4)$$

$$\therefore 4D^2=(\rho'_1-\rho_1)(\rho''_1+\rho'''_1)=(\rho''_2-\rho_2)(\rho'_2+\rho'''_2)=(\rho'''_3-\rho_3)(\rho'_3+\rho''_3)\ldots.(5)$$

By (1) and (2),

$$\left.\begin{array}{l}
\rho'_1\rho''_2\rho'''_3=4D^2r\\
\rho_1\rho'''_2\rho''_3=4D^2r_1\\
\rho'''_1\rho_2\rho'_3=4D^2r_2\\
\rho''_1\rho'_2\rho_3=4D^2r_3
\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From (2, 3), $\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}+\dfrac{1}{\rho_3}=\dfrac{1}{r}(\cos^2\tfrac{1}{2}A+\cos^2\tfrac{1}{2}B+\cos^2\tfrac{1}{2}C)=\dfrac{2}{r}+\dfrac{1}{D}$

$$\left.\begin{array}{l}
\therefore \dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}+\dfrac{1}{\rho_3}=\dfrac{2}{r}+\dfrac{1}{D}\\
\dfrac{1}{\rho'_1}+\dfrac{1}{\rho'_2}+\dfrac{1}{\rho'_3}=\dfrac{2}{r_1}-\dfrac{1}{D}\\
\dfrac{1}{\rho''_1}+\dfrac{1}{\rho''_2}+\dfrac{1}{\rho''_3}=\dfrac{2}{r_2}-\dfrac{1}{D}\\
\dfrac{1}{\rho'''_1}+\dfrac{1}{\rho'''_2}+\dfrac{1}{\rho'''_3}=\dfrac{2}{r_3}-\dfrac{1}{D}
\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

By (1), $\frac{1}{\rho'_1} = \frac{\cos^2\frac{1}{2}A.\cot\frac{1}{2}A}{2D.\cos\frac{1}{2}A.\cos\frac{1}{2}B.\cos\frac{1}{2}C} = \frac{\cot\frac{1}{2}A - \frac{1}{2}\sin A}{2D\cos\frac{1}{2}A.\cos\frac{1}{2}B.\cos\frac{1}{2}}$

Hence, $\frac{1}{\rho'_1} + \frac{1}{\rho''_2} + \frac{1}{\rho'''_3} = \frac{\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C - \frac{1}{2}(\sin A + \sin B\cdot}{2D.\cos\frac{1}{2}A.\cos\frac{1}{2}B\cos\frac{1}{2}C}$

But it is easily shown that

$$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A.\cot\tfrac{1}{2}B.\cot\tfrac{1}{2}C,$$

and $\quad \sin A + \sin B + \sin C = 4\cos\frac{1}{2}A.\cos\frac{1}{2}B.\cos\frac{1}{2}C$;

$$\therefore \frac{1}{\rho'_1} + \frac{1}{\rho''_2} + \frac{1}{\rho'''_3} = \frac{1}{2D.\sin\frac{1}{2}A.\sin\frac{1}{2}B.\sin\frac{1}{2}C} - \frac{1}{D} = \frac{1}{r} - \frac{1}{D}, 1$$

$$\left.\begin{aligned} \therefore \frac{1}{\rho'_1} + \frac{1}{\rho''_2} + \frac{1}{\rho'''_3} &= \frac{1}{r} - \frac{1}{D} \\ \frac{1}{\rho_1} - \frac{1}{\rho'''_2} - \frac{1}{\rho''_3} &= \frac{1}{r_1} + \frac{1}{D} \\ \frac{1}{\rho_2} - \frac{1}{\rho'''_1} - \frac{1}{\rho'_3} &= \frac{1}{r_2} + \frac{1}{D} \\ \frac{1}{\rho_3} - \frac{1}{\rho''_1} - \frac{1}{\rho'_2} &= \frac{1}{r_3} + \frac{1}{D} \end{aligned}\right\} \cdots\cdots\cdots$$

Bearing in mind that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, we readily find from (8) that

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = \frac{1}{\rho'_1} + \frac{1}{\rho'_2} + \frac{1}{\rho'_3} + \frac{1}{\rho''_1} + \frac{1}{\rho''_2} + \frac{1}{\rho''_3} + \frac{1}{\rho'''_1} + \frac{1}{\rho'''_2} + \cdot$$

and

$$\frac{5}{\rho'_1} + \frac{5}{\rho''_2} + \frac{5}{\rho'''_3} = \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \frac{3}{\rho'_2} + \frac{3}{\rho'_3} + \frac{3}{\rho''_1} + \frac{3}{\rho''_3} + \frac{3}{\rho}$$

Some other relations may also be deduced from (7) and (8).

Since $\frac{2}{p_1} = \frac{1}{r} - \frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{r_3}$ (Horæ Geom., Diary for 1843, p.

get from (3), $\quad \frac{1}{\rho_1} - \frac{1}{\rho'_1} = \frac{2\cos^2\frac{1}{2}A}{p_1}$ and $\frac{1}{\rho''_1} + \frac{1}{\rho'''_1} = \frac{2\sin^2\frac{1}{2}A}{p_1}$.

Hence $\frac{1}{\rho_1} - \frac{1}{\rho'_1} + \frac{1}{\rho''_1} + \frac{1}{\rho'''_1} = \frac{2}{p_1}$;

$$\left.\begin{aligned} \therefore \frac{1}{\rho_1} - \frac{1}{\rho'_1} + \frac{1}{\rho''_1} + \frac{1}{\rho'''_1} &= \frac{2}{p_1} \\ \frac{1}{\rho_2} + \frac{1}{\rho'_2} - \frac{1}{\rho''_2} + \frac{1}{\rho'''_2} &= \frac{2}{p_2} \\ \frac{1}{\rho_3} + \frac{1}{\rho'_3} + \frac{1}{\rho'_3} - \frac{1}{\rho'''_3} &= \frac{2}{p_3} \end{aligned}\right\} \cdots\cdots\cdots$$

The equations (3) suggest a very simple geometrical method of the centres R_1, R'_1 *etc.* Draw OM and O_1M' perpendicular to A

O_2M' and O_3M'' perpendicular to O_2O_3 meeting AC in M, M′, M″, M‴, then perpendiculars to AC from M and M′ meet AO_1 in R_1 and R_1' and those from M″ and M‴ meet O_2O_3 in R''_1 and R'''_1. For since r is the perpendicular to AC from O, it is easily seen that $r = OM\cos\frac{1}{2}A = MR_1\cos^2\frac{1}{2}A$; hence by (3), $\rho_1 = MR_1$, and so for the others.

[SECOND SOLUTION.—*Mr. Thomas Dobson, Totteridge, Herts; and similarly by Mr. S. Bills, Hawton.*]

Let O be the centre of the circle circumscribing the triangle ABC; and P the centre of the circle touching AB, AC, and the circle (O).

Join A, O, P; and draw OD perpendicular to AP. Put angle $A=2\alpha$; $B=2\beta$; $C=2\gamma$; and radius of circle $(P)=r'$.

The line AP evidently bisects the angle A, and if O fall within the angle BAP, angle $AOD=\alpha+2\beta$; $\therefore$ $AD=R\sin(\alpha+2\beta)$, $OD=R\cos(\alpha+2\beta)$.

Since AB touches the circle (P), $AP=r'\operatorname{cosec}\alpha$; and because the circles touch externally, $OP=r'+R$.

Now, $DP=AP-AD=r'\operatorname{cosec}\alpha-R\sin(\alpha+2\beta)$; and $OP^2=OD^2+DP^2$; or, $(r'+R)^2=R^2\cos^2(\alpha+2\beta)+\{r'\operatorname{cosec}\alpha-R\sin(\alpha+2\beta)\}^2$; which, after effecting a few obvious reductions, gives

$$r'=2R.\tan^2\alpha\,\{1+\sin(\alpha+2\beta).\operatorname{cosec}\alpha\}.$$

$$\text{But } 2R.\tan^2\alpha=\frac{a.\tan^2\alpha}{\sin2\alpha}=\frac{a.\sin\alpha}{2\cos^3\alpha}=\frac{a}{2}\tan\alpha.\sec^2\alpha;$$

and $1+\sin(\alpha+2\beta)\operatorname{cosec}\alpha=1+\cos2\beta+\cot\alpha.\sin2\beta$

$$=2\cos\beta(\cos\beta+\cot\alpha.\sin\beta)=\frac{2\cos\beta}{\sin\alpha}.\sin(\alpha+\beta)=\frac{2\cos\beta\cos\gamma}{\sin\alpha}=\frac{2s}{a}.$$

Hence, by substitution,

$$r'=s.\tan\alpha.\sec^2\alpha.$$

The radii (r'', r''') of the corresponding circles opposite to the angles B and C are found in a similar way to be

$$r''=s.\tan\beta.\sec^2\beta;\ r'''=s.\tan\gamma.\sec^2\gamma.$$

The positions of the centres P, P_1, P_2 of the circles (r'), (r''), (r'''), are determined by the relations

$$AP=r'\operatorname{cosec}\alpha=\alpha';\ BP_1=r''\operatorname{cosec}\beta=\beta'';\ CP_2=r'''\operatorname{cosec}\gamma=\gamma'''.$$

Cor. 1. The centre (O_1) of the escribed circle (r_1) being likewise in AP, $AO_1=\alpha_1=r_1\operatorname{cosec}\alpha$; hence, $\dfrac{r'}{r_1}=\dfrac{\alpha'}{\alpha_1}$.

$$\text{Similarly,}\qquad \frac{r''}{r_2}=\frac{\beta''}{\beta_2};\ \frac{r'''}{r_3}=\frac{\gamma'''}{\gamma_3}.$$

Cor. 2. If Δ denote the triangle ABC; and the trigonometrical functions of α, β, γ, be expressed in terms of a, b, c; then

$$r'=\frac{bc.\Delta}{s(s-a)^2};\quad r''=\frac{ac.\Delta}{s(s-b)^2};\quad r'''=\frac{ab.\Delta}{s(s-c)^2};$$

$$\therefore\ r'r''r'''=\frac{a^2b^2c^2\Delta^3}{s^3(s-a)^2(s-b)^2(s-c)^2}=\frac{a^2b^2c^2}{s.\Delta}=\frac{a^2b^2c^2}{r_1r_2r_3};$$

and $$r'r''r'''r_1r_2r_3=a^2b^2c^2.$$

Cor. 3. By taking the reciprocals of r', r'', r''', we get

$$\frac{1}{r'}:\frac{1}{r''}:\frac{1}{r'''}::a(s-a)^2:b(s-b)^2:c(s-c)^2.$$

Mr. Bills, after giving the following values of the radii (r'_1, r''_2, r'''_3) of the circles which touch *(externally)* the circumscribing circle, and AB, AC; AB, BC; AC, BC; respectively, viz.

$$r'_1=(s-a)\tan\tfrac{1}{2}A\sec^2\tfrac{1}{2}A,$$
$$r''_2=(s-b)\tan\tfrac{1}{2}B\sec^2\tfrac{1}{2}B,$$
$$r'''_3=(s-c)\tan\tfrac{1}{2}C\sec^2\tfrac{1}{2}C;$$

deduces also the subsequent relations:

$$r'=r_1\sec^2\tfrac{1}{2}A;\quad r'_1=r\sec^2\tfrac{1}{2}A;$$
$$r''=r_2\sec^2\tfrac{1}{2}B;\quad r''_2=r\sec^2\tfrac{1}{2}B;$$
$$r'''=r_3\sec^2\tfrac{1}{2}C;\quad r'''_3=r\sec^2\tfrac{1}{2}C;$$

where r, r_1, r_2, r_3 are the radii of the circles of contact.

Good solutions were also sent by Messrs. John Laws, Newcastle on Tyne; W. H. Levy, Shalbourne; James Reid, Prior's Salford; and G. B. I. Williamson, Hapstone, Durham.

XX.—*Mr. R. H. Wright, London.*

In an inverted cycloid, if δ represent the difference of the times of ascent and descent of each complete oscillation of a body moving in it in a medium whose resistance $=2kv$; show that

$$\delta=\frac{2\sqrt{l}}{\sqrt{g-lk^2}}\sin^{-1}k\sqrt{\frac{l}{g}}.$$

[First Solution.—*Mr. Septimus Tebay, Preston.*]

Let l denote the length of the semi-arc of the cycloid, and $a-s$ the space described by the body at any time t, s being measured from the vertex. Then the resolved portion of gravity in the direction of the curve being $\frac{s}{l}g$, we shall have

$$\frac{dv}{dt}=\frac{s}{l}g-2kv.$$

But, $$v=\frac{d(a-s)}{dt}=-\frac{ds}{dt};$$ and by substitution we have

$$-\frac{d^2s}{dt^2}=\frac{s}{l}g+2k.\frac{ds}{dt},\ \text{or}\ \frac{d^2s}{dt^2}+2k\frac{ds}{dt}+\frac{s}{l}g=0\ \ldots\ldots\ (1)$$

To integrate (1), let $s=ce^{mt}+c'e^{m't}$; then we have

$$\frac{ds}{dt}=mce^{mt}+m'c'e^{m't};\ \frac{d^2s}{dt^2}=m^2ce^{mt}+m'^2c'e^{m't};$$

hence, by substitution in (1), and arranging the terms, we obtain

$$ce^{mt}\left(m^2+2km+\frac{g}{l}\right)+c'e^{m't}\left(m'^2+2km'+\frac{g}{l}\right)=0\dots\dots\dots(2)$$

Now if m and m' are the roots of the quadratic $m^2+2km+\frac{g}{l}=0$, it is evident that equation (2) will be verified by the substitution of these values for m and m'. Resolving the quadratic, we get

$$m=-k+h\sqrt{-1}, \text{ and } m'=-k-h\sqrt{-1},$$

where $h^2=\frac{g}{l}-k^2$; hence the integral becomes

$$s=ce^{-kt+ht\sqrt{-1}}+c'e^{-kt-ht\sqrt{-1}}$$

$$=ce^{-kt}\times e^{ht\sqrt{-1}}+c'e^{-kt}\times e^{-ht\sqrt{-1}}$$

$$=e^{-kt}\left(ce^{ht\sqrt{-1}}+c'e^{-ht\sqrt{-1}}\right).$$

But $e^{ht\sqrt{-1}}=\cos ht+\sqrt{-1}\sin ht$; $e^{-ht\sqrt{-1}}=\cos ht-\sqrt{-1}\sin ht$;

$\therefore\ s=e^{-kt}\{(c+c')\cos ht+(c-c')\sqrt{-1}\sin ht\}=e^{-kt}(\mathrm{C}\cos ht+\mathrm{C}'\sin ht)$;

To determine the constants, let $t=0$, then $s=a$, and $\frac{ds}{dt}=0$. These conditions give $\mathrm{C}=a$, and $\mathrm{C}'=\frac{k}{h}a$; hence the integral is

$$s=\frac{ae^{-kt}}{h}(h\cos ht+k\sin ht)\dots\dots\dots\dots(3)$$

From (3) we derive

$$v=\frac{ds}{dt}=\frac{age^{-kt}}{hl}\sin ht;$$

and when $v=0$, we have $\sin ht=0\ \therefore\ ht=\pi$; hence if T be the time of an oscillation, we have

$$\mathrm{T}=\frac{\pi}{h}.$$

Also, when $s=0$, we have $h\cos ht+k\sin ht=0$, and from this we get

$$\sin ht=h\sqrt{\frac{l}{g}}, \text{ or } t=\frac{1}{h}\sin^{-1}h\sqrt{\frac{l}{g}}=\text{time of descent};$$

$$\therefore\ \delta=\mathrm{T}-2t=\frac{\pi-2\sin^{-1}h\sqrt{\frac{l}{g}}}{h}=\frac{2}{h}\left(\frac{\pi}{2}-\sin^{-1}h\sqrt{\frac{l}{g}}\right)$$

$$=\frac{2}{h}\sin^{-1}k\sqrt{\frac{l}{g}}=\frac{2\sqrt{l}}{\sqrt{g-lk^2}}\sin^{-1}k\sqrt{\frac{l}{g}}.$$

[SECOND SOLUTION.—*Mr. R. H. Wright, the proposer.*]

By Book II., chap. iii., prob. 1., Whewell's Dynamics, if s be the arc AP from the lowest point A, force at P $=\frac{s}{l}g$, where A C $= l$, resistance $= 2kv$, $h^2=\frac{g}{l}-k^2$, and t the time of an oscillation, it is shown that $t=\frac{\pi}{h}$.

Let $t_1=$ time of descent to lowest point, and $t_2 =$ time of ascent from lowest point; then $\delta = t_2 - t_1$, since t_2 is greater than t_1. Now when the body arrives at the lowest point, the velocity is a minimum, and the accelerating force zero (ibid, cor. 1.); hence,

$$\tan ht = \frac{h}{k}, \quad \text{or} \quad t_1 = \frac{1}{h}\tan^{-1}\frac{h}{k}$$

$$\therefore t_2 = \frac{\pi}{h} - \frac{1}{h}\tan^{-1}\frac{h}{k},$$

hence, $$\delta = t_2 - t_1 = \frac{1}{h}\left(\pi - 2\tan^{-1}\frac{h}{k}\right), \text{ and therefore}$$

$$\tan^{-1}\frac{h}{k} = \frac{\pi}{2} - \frac{h\delta}{2}, \quad \cot^{-1}\frac{h}{k} = \frac{h\delta}{2};$$

$$\therefore \quad \frac{h}{k} = \frac{\cos\frac{h\delta}{2}}{\sin\frac{h\delta}{2}}, \text{ or } \sin^2\frac{h\delta}{2} = \frac{k^2}{h^2+k^2};$$

$$\therefore \quad \delta = \frac{2}{h}\sin^{-1}\frac{k}{\sqrt{h^2+k^2}} = \frac{2\sqrt{l}}{\sqrt{g-lk^2}}\sin^{-1}k\sqrt{\frac{l}{g}}.$$

[THIRD SOLUTION.—*By A.*]

It is a well known property of the cycloid, that the cosine of the angle which the tangent to it at any point makes with the vertical is $\frac{s}{l}$, where s is the length of the arc between that point and the lowest point. The equation of motion is therefore

$$\frac{d^2s}{dt^2} + 2k\frac{ds}{dt} + \frac{g}{l}s = 0.$$

The complete integral of this equation is (Lacroix, Elem. Treat. Art. 308-310.)

$$s = e^{-kt}(c\cos ht + c'\sin ht) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where $h^2=\frac{g}{l} - k^2$ and c, c' are constants depending upon any simultaneous values of t, s and $\frac{ds}{dt}$.

To simplify the equation, let $t=0$, then the velocity, that is $\frac{ds}{dt}$, must also be zero. From these conditions it is easy to find that

$$\frac{c}{c'}=\frac{h}{k}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and that for any time t,

$$\frac{ds}{dt}=-(ch+c'k)e^{-kt}\sin ht\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

By making $s=0$ in (1) we find $c\cos ht+c'\sin ht=0$, from which we get

$$\sin^2 ht=\frac{c^2}{c^2+c'^2}=\frac{h^2}{h^2+k^2}=h^2\frac{l}{g};$$

$$\therefore\quad \sin ht=h\sqrt{\frac{l}{g}},\text{ or } t=\frac{1}{h}\sin^{-1}h\sqrt{\frac{l}{g}}.$$

Taking the least value of t derived from this formula, we have the time of descent.

At the end of a complete oscillation, it is plain that $\frac{ds}{dt}=0$, and substituting this value in (3), we have $\sin ht=0$. The least value of t, exclusive of zero, derived from this formula, is evidently the time of a complete oscillation; hence

$$ht=\pi,\text{ or } t=\frac{\pi}{h}.$$

The time of ascent is equal to the time of a complete oscillation *minus* the time of descent, and therefore it is

$$\frac{\pi}{h}-\frac{1}{h}\sin^{-1}h\sqrt{\frac{l}{g}}.$$

From this it follows that

$$\delta=\left\{\frac{\pi}{h}-\frac{1}{h}\sin^{-1}h\sqrt{\frac{l}{g}}\right\}-\frac{1}{h}\sin^{-1}h\sqrt{\frac{l}{g}}$$

$$=\frac{2}{h}\left\{\frac{\pi}{2}-\sin^{-1}h\sqrt{\frac{l}{g}}\right\}=\frac{2}{h}\cos^{-1}h\sqrt{\frac{l}{g}}$$

$$=\frac{2}{h}\sin^{-1}k\sqrt{\frac{l}{g}}=\frac{2\sqrt{l}}{\sqrt{g-lk^2}}\sin^{-1}k\sqrt{\frac{l}{g}}.$$

Scholium.—The integration of differential equations is a subject of considerable difficulty, and the labours of analysts have not hitherto removed the obscurity in which it is enveloped. There is, however, a large class of differential equations, termed *Linear Equations*, of great importance in the higher departments of Mathematics, which are integrable by one method, and the differential equation in this exercise is an instance. The separation of the symbols of operation from those of quantity contributes to the elegance of the method of integration, but for further information the reader is referred to GREGORY's Examples on the Differential and Integral Calculus, chap. iv., p. 287. R.

XXI.—*Mr. James Dalmahoy, Edinburgh.*

Let a tangent be drawn to a conic section at either of its vertices, and likewise its circle of curvature at the same point; then any line drawn to touch one of the curves and to cut the other and the tangent, will be harmonically divided in the points of contact and intersection.

[SOLUTION.—*Mr. St. Andrew St. John, Gent. Cadet, R.M.A.*]

Let us conceive another conic section referred to the same rectangular axes and origin as the proposed one, (the origin being the vertex, and the axes, the principal diameters) and having with it a contact of the second order: then that the first and second differential coefficients in the two equations may be equal, it is readily found that the equations of these two curves will be

$$y^2 = 2mx + nx^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$y^2 = 2mx + px^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now denote the point of contact of the inner curve (here supposed to be the second) with a tangent which cuts the other by x_1y_1, the equation of that tangent is

$$y_1y = m\ (x + x_1) + px_1x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Substituting for y from (1), and for y_1 from (2), the point x_1y_1, being in that curve, we shall get the equation containing the abscisses x of the two points of intersection. This equation is

$$\{m^2+(p-n)(2mx_1+px^2_1)\}x^2-2m^2x_1x+m^2x^2_1=0 \ldots\ldots\ldots (4)$$

Now if x_2, x_3 be the roots of this equation, we shall obviously have

$$x_2 + x_3 = \frac{2m^2x_1}{m^2+(p-n)\ (2mx_1+px^2_1)} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$x_2x_3 = \frac{m^2x_1^{\ 2}}{m^2+(p-n)\ (2mx_1+px^2_1)} \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and hence by division of (5) by (6) we get

$$\frac{x_2+x_3}{x_2x_3} = \frac{1}{x_2} + \frac{1}{x_3} = \frac{2m^2x_1}{m^2x^2_1} = \frac{2}{x_1} \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

which is a well known property of a line divided harmonically; and shews that the axis of x is divided harmonically at the distances x_1, x_2, x_3 from the origin.

Also the corresponding segments of the tangent itself are $x_1 \sec\beta$, $x_2 \sec\beta$, $x_3\sec\beta$, when β is the inclination of the tangent to the axis of x: and since these have the same ratios as x_1, x_2, x_3, the property is fully proved in a form somewhat more general than was proposed in the exercise.

Mr. Weddle's solution was very similar to the above; and correct solutions were received from Messrs. Bills, Dobson, and Theta. Mr. Dalmahoy's own geometrical solution was very elegant; but we retain it at present, as the subject will be further discussed and some general properties of the circle of curvature will be given on a future occasion.

XXIII.—*Mr. Woolhouse, Editor of the Lady's and Gentleman's Diary.*

According to the usual functional notation

$\sin^2 x$ denoting sin. of $\sin x$,

$\sin^3 x$ „ sin of $\sin^2 x$,

. . . .

it is required to find the limiting value of the fraction

$$\frac{x^{n-1}\sin^n x-(\sin x)^n}{x^{n+4}}$$

when $x=0$.

[First Solution.—*Mr. G. W. Hearn, Royal Military College, Sandhurst.*]

$$\text{Let } \sin^n x = a_n + \frac{b_n}{1}x + \frac{c_n}{1\cdot 2}x^2 + \frac{d_n}{1\cdot 2\cdot 3}x^3 + \frac{e_n}{1\cdot 2\cdot 3\cdot 4}x^4 + \ldots\ldots,$$

in which, since $\sin^n x$ cannot involve even powers, we know *a priori* that a_n, c_n, e_n, *etc.*, are severally $=0$, and are merely retained in order to find equations of differences which are deduced by differentiation as follows:

$$\sin^n x = \sin(\sin^{n-1}x)\,;\ \therefore\ a_n = \sin a_{n-1}\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating successively, we have

$$b_n = \cos a_{n-1}.\,b_{n-1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$c_n = -a_n\,b^2_{n-1}+\cos a_{n-1}.c_{n-1}\ldots\ldots\ldots\ldots\ldots(3)$$

$$d_n = -b_n\,b^2_{n-1}-3a_n\,b_{n-1}\,c_{n-1}+\cos a_{n-1}\,d_{n-1}\ldots(4)$$

$$e_n = -c_n\,b^2_{n-1}-5b_n\,b_{n-1}\,c_{n-1}-3a_n\,c^2_{n-1}-4a_n\,b_{n-1}d_{n-1}+\cos a_{n-1}\,e_{n-1}\ \ldots(5)$$

$$f_n = -d_n\,b^2_{n-1}-7c_n\,b_{n-1}c_{n-1}-8b_n\,c^2_{n-1}-9b_n\,b_{n-1}d_{n-1}-10a_n\,c_{n-1}d_{n-1}$$
$$-5a_n\,b_{n-1}c_{n-1}+\cos a_{n-1}f_{n-1}\ldots\ldots\ldots(6)$$

Hence in (2), since $a_{n-1}=0$, we have $b_n = b_{n-1}$, or $\Delta\,b_n = 0\ \therefore b_n = \text{const.} = b_1 = 1$; therefore in (4), $d_n = -1+d_{n-1}$, or $\Delta d_n = -1$; hence $d_n = -n$.

Also in (6),

$f_n - f_{n-1} = n + 9(n-1)$, or $\Delta f_n = 10n+1$; therefore $f_n = n\,(5n-4)$;

$$\text{Hence, } \sin^n x = x - \frac{n}{1\cdot 2\cdot 3}x^3 + \frac{n(5n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 + \ldots\ldots\ldots\ldots$$

Also, by the usual process,

$$(\sin x)^n = x^n - \frac{n}{1\cdot 2\cdot 3}x^{n+2} + \frac{n(5n-2)}{3\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}x^{n+4} -, \textit{etc.}$$

Hence the first term in $x^{n-1}\sin^n x-(\sin x)^n$ is $=\dfrac{n(n-1)}{36}x^{n+4}$, the next involving higher powers of x, so that the limit of

$$\frac{x^{n-1}\sin^n x-(\sin x)^n}{x^{n+4}} = \frac{n(n-1)}{36}, \text{ when } x=0.$$

Scholium.—If the denominator of the proposed expression were x^5 instead of x^{n+4}, the first term of the expansion of the fraction would be $\dfrac{n(n-1)}{36}x^{n-1}$, and hence the limiting value of the fraction, when $x=0$, would be zero.

[SECOND SOLUTION.—*By A.*]

Assume $\sin^n x = x + v_n x^3 + u_n x^5 + \ldots etc.$, it being evident that there are no even powers; then we have

$$\sin^{n+1} x = x + v_{n+1} x^3 + u_{n+1} x^5 + \ldots etc.$$

Now $\sin^{n+1} x = \sin(\sin^n x) = \sin^n x - \frac{(\sin^n x)^3}{1\cdot2\cdot3} + \frac{(\sin^n x)^5}{1\cdot2\cdot3\cdot4\cdot5} - \ldots etc.$

$$= x + \left(v_n - \frac{1}{1\cdot2\cdot3}\right) x^3 + \left(u_n - \frac{3v_n}{1\cdot2\cdot3} + \frac{1}{1\cdot2\cdot3\cdot4\cdot5}\right) x^5 + \ldots etc.$$

Hence, identically,

$$v_{n+1} = v_n - \frac{1}{1\cdot2\cdot3} \ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots (1)$$

$$\text{and } u_{n+1} = u_n - \frac{3v_n}{1\cdot2\cdot3} + \frac{1}{1\cdot2\cdot3\cdot4\cdot5} \ldots\ldots\ldots\ldots (2)$$

From the first we have

$$\Delta v_n = -\frac{1}{1\cdot2\cdot3},$$

and by integration, $v_n = -\frac{n}{1\cdot2\cdot3} + \text{C}$. But v_1 is known to be $-\frac{1}{1\cdot2\cdot3}$; and hence $\text{C} = 0$; therefore

$$v_n = -\frac{n}{1\cdot2.3}.$$

Substituting this value of v_n in (2), we have, after reduction,

$$\Delta u_n = \frac{10n+1}{1\cdot2\cdot3\cdot4\cdot5}.$$

And by integration, $\qquad u_n = \frac{n(5n-4)}{1\cdot2\cdot3\cdot4\cdot5} + \text{C}.$

But $u_1 = \frac{1}{1\cdot2\cdot3\cdot4\cdot5} = \frac{1}{1\cdot2\cdot3\cdot4\cdot5} + \text{C}$, so that $\text{C} = 0$; hence we have

$$\sin^n x = x - \frac{n}{1\cdot2\cdot3} x^3 + \frac{n(5n-4)}{1\cdot2\cdot3\cdot4\cdot5} x^5 - \ldots\ldots$$

Again, $\qquad (\sin x)^n = \left(x - \frac{x^3}{1\cdot2\cdot3} + \frac{x^5}{1\cdot2\cdot3\cdot4\cdot5} - \ldots\ldots\right)^n$

$$= x^n - \frac{n}{1\cdot2\cdot3} x^{n+2} + \frac{n(5n-2)}{3\cdot1\cdot2\cdot3\cdot4\cdot5} x^{n+4} - \ldots.$$

Hence, $\qquad \frac{x^{n-1}\sin^n x - (\sin x)^n}{x^{n+4}} = \frac{n(n-1)}{36} + \text{A}x^2 + \ldots\ldots,$

and therefore the limiting value, when $x = 0$, is $\frac{n(n-1)}{36}$.

[THIRD SOLUTION.—*Mr. Weddle.*]

Assume, $(\sin x)^n = x^n - \phi_1(n).\, x^{n+2} + \phi_2(n).\, x^{n+4} - etc. \ldots\ldots\ldots\ldots ($

Now $\quad \sin x = x - \frac{1}{2\cdot3}\cdot x^3 + \frac{1}{2\cdot3\cdot4\cdot5}\cdot x^5 - \frac{1}{2\cdot3..7}\cdot x^7 + etc. \ldots\ldots$

Multiply (1) by (2), then $(\sin x)^{n+1} =$

$$-\left\{\phi_1(n) + \frac{1}{2\cdot3}\right\} x^{n+3} + \left\{\phi_2(n) + \frac{1}{2\cdot3}\cdot\phi_1(n) + \frac{1}{2\cdot3\cdot4\cdot5}\right\} x^{n+5} + etc.$$

But, by the notation,

$(\sin x)^{n+1} = x^{n+1} - \phi_1(n+1).\ x^{n+3} + \phi_2(n+1).\ x^{n+5} - etc.$ ....(4)

By comparing the coefficients of (3) and (4) we get

$$\Delta\,\phi_1(n) = \frac{1}{2\cdot3}$$

$$\Delta\,\phi_2(n) = \frac{1}{2\cdot3}.\ \phi_1(n) + \frac{1}{2\cdot3\cdot4\cdot5}$$

$$\Delta\,\phi_3(n) = \frac{1}{2\cdot3}.\ \phi_2(n) + \frac{1}{2\cdot3\cdot4\cdot5}\phi_1(n) + \frac{1}{2\cdot3..7},\ etc.$$

Hence, $\phi_1(n) = \frac{n}{2\cdot3} = \frac{n}{6}$

$$\therefore\ \Delta\phi_2(n) = \frac{1}{2\cdot3}.\ \frac{n}{6} + \frac{1}{2\cdot3\cdot4\cdot5}$$

$$\therefore\ \phi_2(n) = \frac{1}{72}.\ n(n-1) + \frac{n}{2\cdot3\cdot4\cdot5} = \frac{n(5n-2)}{360}.$$

Hence, $\Delta\phi_3(n) = \frac{1}{2.3}.\ \phi_2(n) + \frac{1}{2\cdot3\cdot4\cdot5}.\phi_1(n) + \frac{1}{2\cdot3..7} = \frac{1}{432}n(n-1)$

$$+ \frac{1}{3}.\ \frac{n}{2\cdot3\cdot4.5} + \frac{1}{2\cdot3..7}.$$

$$\therefore\ \phi_3(n) = \frac{1}{1296}n(n-1)(n-2) + \frac{1}{2\cdot3}.\ \frac{n(n-1)}{2\cdot3\cdot4\cdot5} + \frac{n}{2.3..7}$$

$$= \frac{n(35n^2-42n+16)}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot9},\ etc.$$

The constants being each $=0$, for $\phi_1(1) = \frac{1}{2\cdot3}$, $\phi_2(1) = \frac{1}{2\cdot3\cdot4\cdot5}$, *etc.*

$$.\ (\sin x)^n = x^n - \frac{n}{6}.\ x^{n+2} + \frac{n(5n-2)}{360}.\ x^{n+4} - \frac{n(35n^2-42n+16)}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot9}.x^{n+6} +$$

etc. ....(5)

Again, let $\sin^n x = x - f_1(n).\ x^3 + f_2(n).x^5 - f_3(n).\ x^7 + etc.$ .....(6)

$$\therefore\ \sin^{n+1}x = \sin\{x - f_1(n).\ x^3 + f_2(n).\ x^5 - f_3(n).\ x^7 + etc.\}$$

$$= x - f_1(n)\ x^3 + f_2(n).\ x^5 - f_3(n).\ x^7 + etc..$$

$$- \frac{1}{2\cdot3}\{x - f_1(n).x^3 + f_2(n).\ x^5 - etc.\}^3$$

$$+ \frac{1}{2\cdot3\cdot4\cdot5}\{x - f_1(n).\ x^3 + etc.\}^5$$

$$- \frac{1}{2\cdot3\cdot4\cdot5\cdot6\cdot7}\{x - etc.\}^7$$

$$+\ etc.$$

or, by reduction,

$$\sin^{n+1}x = x - \left\{f_1(n) + \frac{1}{2\cdot3}\right\}.\ x^3 + \left\{f_2(n) + \tfrac{1}{2}f_1(n) + \frac{1}{2\cdot3\cdot4\cdot5}\right\}x^5$$

$$-\left\{f_3(n) + \tfrac{1}{2}f_2(n) + \tfrac{1}{2}\,\overline{f_1(n)}|^2 + \frac{1}{2\cdot3\cdot4}f_1(n) + \frac{1}{2\cdot3..7}\right\}x^7.$$

$+\ etc.$ ....(7)

Now, $\sin^{n+1}x = x - f_1(n+1).x^3 + f_2(n+1).x^5 - f_3(n+1).x^7 + etc. \ldots\ldots (8)$

By comparing the coefficients of (7, 8), we get

$$\Delta f_1(n) = \frac{1}{2\cdot3}$$

$$\Delta f_2(n) = \tfrac{1}{2} f_1(n) + \frac{1}{2\cdot3\cdot4\cdot5}$$

$$\Delta f_3(n) = \tfrac{1}{2} f_2(n) + \tfrac{1}{2}\{f_1(n)\}^2 + \frac{1}{2\cdot3\cdot4} f_1(n) + \frac{1}{2\cdot3\,..\,7}\text{; } etc.$$

$$\therefore\; f_1(n) = \frac{n}{2\cdot3} = \frac{n}{6}.$$

$$\Delta f_2(n) = \frac{n}{12} + \frac{1}{2\cdot3\cdot4\cdot5}\text{; or } f_2(n) = \frac{n(n-1)}{24} + \frac{n}{2\cdot3\cdot4\cdot5} = \frac{n(5n-4)}{120};$$

$$\therefore\; \Delta f_3(n) = \frac{n(n-1)}{48} + \frac{n}{2\cdot2\cdot3\cdot4\cdot5} + \frac{n^2}{72} + \frac{n}{2\cdot3\cdot4.6} + \frac{1}{2\cdot3\,..\,7}$$

$$= \frac{5n(n-1)}{2\cdot3\cdot4\cdot6} + \frac{n}{2\cdot4\cdot5} + \frac{1}{2\cdot3\,..\,7}$$

$$\therefore f_3(n) = \frac{5n(n-1)(n-2)}{2\cdot4\cdot6\cdot9} + \frac{n(n-1)}{4\cdot4\cdot5} + \frac{n}{2\cdot3\cdot\,..\,7} = \frac{n(175n^2-336n+164)}{2\cdot4\cdot5\cdot6\cdot7\cdot9}$$

etc.

Here the constants $= 0$, for the same reason as before;

$$\therefore\; \sin^n x = x - \frac{n}{6}x^3 + \frac{n(5n-4)}{120}.x^5 - \frac{n(175n^2-336n+164)}{2\cdot4\cdot5\cdot6\cdot7\cdot9}x^7 + etc. \quad \ldots\ldots\ldots (9)$$

From (5) and (9), we have

$$\frac{x^{n-1}\sin^n x - (\sin x)^n}{x^{n+4}} = \frac{n(n-1)}{36} - \frac{n(n-1)(259n-266)}{3\cdot4\cdot5\cdot6\cdot7\cdot9}x^2 + etc. \ldots (10)$$

Hence the limiting value of the proposed fractional expression, when $x = 0$, is obviously $= \dfrac{n(n-1)}{36}$.

Solutions, on different principles, were received from Messrs. Beecroft, Dobson, and J. W. Elliott, Greatham, Stockton-on-Tees.

XXIV.—*Mr. Geo. W. Hearn, R. M. Coll. Sandhurst.*

Let there be three circles in the same plane, then in general *eight* other circles may be drawn touching all the three; let R_n be the radius of one which is touched externally by n of the given circles, and $\Sigma \dfrac{1}{R_n}$ the sum of the reciprocals of the radii of all the circles which are touched externally by n of the given circles; then

$$\frac{1}{R_3} + \Sigma\frac{1}{R_1} = \frac{1}{R_0} + \Sigma\frac{1}{R_2}.$$

[FIRST SOLUTION.—*Mr. Geo. W. Hearn, the proposer.*]

Denote by r_1, r_2, r_3, the radii of the given circles whose centres are A,B,C, and let r be the required radius of the centre circle O supposed to touch the three given circles externally. Let a, b, c denote the distances BC, CA,

AB respectively, and put CAO $= \theta$, and BAO $= \theta'$; then from the triangle AOC we have

$$(r+r_3)^2 = (r+r_1)^2 + b^2 - 2b(r+r_1)\cos\theta,$$

$$\text{or, } \cos\theta = \frac{b^2+(r_1-r_3)(2r+r_1+r_3)}{2b(r+r_1)}.$$

$$\text{Similarly,}\quad \cos\theta' = \frac{c^2+(r_1-r_2)(2r+r_1+r_2)}{2c(r+r_1)};$$

but $\theta+\theta' = \text{A}$, $\therefore\ \cos\theta\cos\theta' - \sin\theta\sin\theta' = \cos\text{A}$; hence

$$\cos^2\theta + \cos^2\theta' - 2\cos\text{A}\cos\theta\cos\theta' = \sin^2\text{A},\ \text{where } \cos\text{A} = \frac{b^2+c^2-a^2}{2bc}.$$

Hence, we are led to the following equation,

$$c^2\{b^2+(r_1-r_3)(2r+r_1+r_3)\}^2 + b^2\{c^2+(r_1-r_2)(2r+r_1+r_2)\}^2$$
$$+(a^2-b^2-c^2)\{b^2+(r_1-r_3)(2r+r_1+r_3)\}\{c^2+(r_1-r_2)(2r+r_1+r_2)\}$$
$$+\{a^2-(b+c)^2\}\{a^2-(b-c)^2\}(r+r_1)^2 = 0,$$

which will be a quadratic for determining r.

In this the values of r will be R_3 and $-\text{R}_0$, because the equation for finding R_0 may be deduced from this by changing the signs of r, r_1, r_2, r_3, by which means an equation will be formed which is the same as that just written, except in the sign of the coefficient of r, so that its roots are R_0 and $-\text{R}_3$.

Now the terms independent of r in the above, involve only even powers of r_1, r_2, r_3, and therefore do not change when any or all of these quantities change sign. Denote then these terms by Q.

The terms which form the coefficient of r are of the first and third orders in r_1, r_2, r_3. Let them be denoted by $-$P, then we will have

$$\frac{1}{\text{R}_3} - \frac{1}{\text{R}_0} = \frac{\text{P}}{\text{Q}}.$$

Now the terms in P are of the powers $\text{A}r_1$, $\text{B}r_1r_2^2$, $\text{C}r_1^3$; hence if in $\frac{1}{\text{R}_3} - \frac{1}{\text{R}_0}$ we have

$$\frac{\text{A}r_1+\text{B}r_1r_2^2+\text{C}r_1^3}{\text{Q}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

we will have in the equations resulting from changing the signs of

$$(r_1, r_2)\qquad \frac{-\text{A}r_1-\text{B}r_1r_2^2-\text{C}r_1^3}{\text{Q}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$(r_1, r_3)\qquad \frac{-\text{A}r_1-\text{B}r_1r_2^2-\text{C}r_1^3}{\text{Q}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$(r_2, r_3)\qquad \frac{\text{A}r_1+\text{B}r_1r_2^2+\text{C}r_1^3}{\text{Q}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

But the sum of these $= 0$, and so of the other terms. Now (2), (3), and (4) occur in $\Sigma\frac{1}{\text{R}_1} - \Sigma\frac{1}{\text{R}_2}$; hence,

$$\frac{1}{\text{R}_3} - \frac{1}{\text{R}_0} + \Sigma\left(\frac{1}{\text{R}_1} - \frac{1}{\text{R}_2}\right) = 0,$$

$$\text{or,}\quad \frac{1}{\text{R}_3} + \Sigma\frac{1}{\text{R}_1} = \frac{1}{\text{R}_0} + \Sigma\frac{1}{\text{R}_2},$$

which establishes the theorem.

Cor. When the given circles are in mutual contact, the three circles in $\Sigma \frac{1}{R_2}$ become evidently identical with the given circles. Again, no circle can be touched externally by only *one* of the circles of such a system. Hence we ought to change the sign of each in $\Sigma \frac{1}{R_1}$, and then they become also identical with the three given circles, so that in this case.

$$\frac{1}{R_3} - \frac{1}{R_0} = 2\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right),$$

provided there be an *enveloping* circle radius R_0, and if there be not, then also we should change the sign of R_0, and call it $-R'_3$, so that

$$\frac{1}{R_3} + \frac{1}{R'_3} = 2\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right).$$

The case in which the three given circles pass through the same point makes $Q = 0$, so that the result is presented under the form $\frac{0}{0}$, and requires a separate discussion. But this case has already been discussed by myself and others in the Lady's and Gentleman's Diary for 1844.

[SECOND SOLUTION.—*Mr. Samuel Bills, Hawton.*]

Let A, B, C be the centres, and r_1, r_2, r_3 the radii of the three given circles; let O be the centre, R the radius, and ρ the *reciprocal* of the radius of a circle which is touched *externally* by the three given circles. Put $BC = a$, $CA = b$, $AB = c$, $BAC = \theta$, $BAO = \theta_1$, $CAO = \theta_2$; then $\theta = \theta_1 + \theta_2$, and hence we readily deduce

$$1 - \cos^2\theta - \cos^2\theta_1 - \cos^2\theta_2 + 2\cos\theta\cos\theta_1\cos\theta_2 = 0 \;\ldots\ldots\ldots (A)$$

But $\cos\theta_1 = \dfrac{c^2+r^2_1-r^2_2+2(r_1-r_2)R}{2c(r_1+R)} = \dfrac{\dfrac{c^2+r^2_1-r^2_2}{2c}\cdot\dfrac{1}{R}+\dfrac{1}{c}(r_1-r_2)}{\dfrac{r_1}{R}+1}$

$$= \frac{f\rho+g(r_1-r_2)}{r_1\rho+1}, \text{ where } f = \frac{c^2+r^2_1-r^2_2}{2c}, \text{ and } g = \frac{1}{c}.$$

Similarly, $\cos\theta_2 = \dfrac{f_1\rho+g_1(r_1-r_3)}{r_1\rho+1}$, where $f_1 = \dfrac{b^2+r^2_1-r^2_3}{2b}$, and $g_1 = \dfrac{1}{b}$.

Substituting these values of $\cos\theta_1$ and $\cos\theta_2$ in equation (A), it is evident by bare inspection that the resulting equation, when reduced, will be of the form

$$\rho^2 - 2(hr_1+h_1r_2+h_2r_3)\rho + h_3 + h_4r_1r_2 + h_5r_1r_3 + h_6r_2r_3 = 0;$$

where h, h_1, h_2, *etc.* are functions of r^2_1, r^2_2, r^2_3, and are hence independent of the signs of these radii. Resolving the last equation for ρ, we have

$$\rho = hr_1 + h_1r_2 + h_2r_3 + (k + k_1r_1r_2 + k_2r_1r_3 + k_3r_2r_3)^{\frac{1}{2}}$$

where k, k_1, k_2, *etc.* are also functions of r^2_1, r^2_2, r^2_3, and do not change their values whether r_1, r_2, r_3 be considered *positive* or *negative*.

Now this investigation will equally apply to the seven other circles of contact, by merely changing the signs of r_1, r_2, r_3 in the cases of internal contact. Denoting, therefore, the radii of the other seven circles by

$R_1, R_2, \ldots R_7$, or their reciprocals by $\rho_1, \rho_2, \ldots \rho_7$, we have obviously the following system of equations:

$(r_1, r_2, r_3$ positive$)\ldots\rho = \quad hr_1+h_1r_2+h_2r_3+(k+k_1r_1r_2+k_2r_1r_3+k_3r_2r_3)^{\frac{1}{2}}\ldots(1)$

$(r_1$ negative$)\ldots\rho_1= -hr_1+h_1r_2+h_2r_3+(k-k_1r_1r_2-k_2r_1r_3+k_3r_2r_3)^{\frac{1}{2}}\ldots(2)$

$(r_2$ negative$)\ldots\rho_2= \quad hr_1-h_1r_2+h_2r_3+(k-k_1r_1r_2+k_2r_1r_3-k_3r_2r_3)^{\frac{1}{2}}\ldots(3)$

$(r_3$ negative$)\ldots\rho_3= \quad hr_1+h_1r_2-h_2r_3+(k+k_1r_1r_2-k_2r_1r_3-k_3r_2r_3)^{\frac{1}{2}}\ldots(4)$

$(r_1, r_2$ negative$)\ldots\rho_4= -hr_1-h_1r_2+h_2r_3+(k+k_1r_1r_2-k_2r_1r_3-k_3r_2r_3)^{\frac{1}{2}}\ldots(5)$

$(r_1\ r_3$ negative$)\ldots\rho_5= -hr_1+h_1r_2-h_2r_3+(k-k_1r_1r_2+k_2r_1r_3-k_3r_2r_3)^{\frac{1}{2}}\ldots(6)$

$(r_2, r_3$ negative$)\ldots\rho_6= \quad hr_1-h_1r_2-h_2r_3+(k-k_1r_1r_2-k_2r_1r_3+k_3r_2r_3)^{\frac{1}{2}}\ldots(7)$

$(r_1, r_2, r_3$ negative$)\ldots\rho_7= -hr_1-h_1r_2-h_2r_3+(k+k_1r_1r_2+k_2r_1r_3+k_3r_2r_3)^{\frac{1}{2}}\ldots(8)$

Hence it is manifest that

$$\rho+\rho_4+\rho_5+\rho_6=\rho_7+\rho_1+\rho_2+\rho_3$$

$$\text{or, } \frac{1}{R}+\frac{1}{R_4}+\frac{1}{R_5}+\frac{1}{R_6}=\frac{1}{R_7}+\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3};$$

which is the beautiful theorem enunciated. The demonstration rests upon the principle that f, f_1, h, h_1, *etc.*, k, k_1, k_2, *etc.*, represent such functions of the radii r_1, r_2, r_3 as involve only the squares of these quantities, and hence they preserve the same values when these quantities have different signs.

The analogous property when the circles are described on a sphere may be proved in a similar manner, and with almost equal facility. The property for the sphere is

$$\cot R+\cot R_4+\cot R_5+\cot R_6=\cot R_7+\cot R_1+\cot R_2+\cot R_3.$$

[THIRD SOLUTION.—*Mr. Thomas Weddle, Newcastle-on-Tyne.*]

Let A, B, C be the centres and a, β, γ the radii of the three circles; draw CM perpendicular to AB; assume MB, MC for rectangular axes, and denote the points A, B, and C by (a, o), (b, o), and (o, c) respectively. Let x, y be the co-ordinates of the centre, and R the radius, of one of the circles touching the three given ones; then if this circle touches all the other three externally, R will be found by eliminating x, y from the equations (A). But if the circle radius R touch that whose radius is a, β, or γ externally, and the other two circles internally, R will be found from (B), (C) or (D).

$$\left.\begin{array}{l}(1)\ldots(x-a)^2+y^2=(a+R)^2\\(2)\ldots(x-b)^2+y^2=(\beta+R)^2\\(3)\ldots x^2+(y-c)^2=(\gamma+R)^2\end{array}\right\}\ldots(A)\qquad\left.\begin{array}{l}(1)\ldots(x-a)^2+y^2=(a-R)^2\\(2)\ldots(x-b)^2+y^2=(\beta+R)^2\\(3)\ldots x^2+(y-c)^2=(\gamma-R)^2\end{array}\right\}\ldots(C)$$

$$\left.\begin{array}{l}(1)\ldots(x-a)^2+y^2=(a+R)^2\\(2)\ldots(x-b)^2+y^2=(\beta-R)^2\\(3)\ldots x^2+(y-c)^2=(\gamma-R)^2\end{array}\right\}\ldots(B)\qquad\left.\begin{array}{l}(1)\ldots(x-a)^2+y^2=(a-R)^2\\(2)\ldots(x-b)^2+y^2=(\beta-R)^2\\(3)\ldots x^2+(y-c)^2=(\gamma=R)^2\end{array}\right\}\ldots(D)$$

But if the circle radius R touch none of the circles externally, or if it touch only two of them externally, R will be found from some of the preceding equations after writing $-$R for R. Hence in every case, the radius R will be found from some of the above sets of equations, provided we con-

sider a radius negative, when its circle touches either none of the circles externally or only two of them.

Let r, ρ; r_1, ρ_1; r_2, ρ_2; and r_3, ρ_3 be the radii whose values are derived from (A), (B), (C), and (D) respectively, the signs of these radii being estimated as above.

Deduct (2) from (1), and we get a result which may be put under the form

$$x - a = m(\alpha - \beta)\,R - M \qquad \text{(4)}$$

where m is independent of α, β, γ, and M involves only even powers of them. Deduct (3) from (2), and eliminate x by means of (4), we thus have a result of the form

$$y = (n\alpha + p\beta + q\gamma)\,R - N \qquad \text{(5)}$$

n, p, q being similar to m, and N to M. Substitute (4) and (5) in (1), reduce and put δ for the co-efficient of R^2;

$$(M^2 + N^2 - a^2) - 2\{Mm(\alpha - \beta) + N(m\alpha + p\beta + q\gamma) + a\}R + \delta R^2 = 0.$$

Divide by $M^2 + N^2 - a^2$, we thus have a result of the form

$$1 - \{P\alpha + Q\beta + S\gamma\}R + \delta' R^2 = 0 \qquad \text{(6)}$$

where P, Q, S obviously involve only even powers of α, β, γ.

Now, the roots of (6) are γ and ρ; hence, by the theory of equations,

$$\frac{1}{r} + \frac{1}{\rho} = P\alpha + Q\beta + S\gamma \qquad \text{(7)}$$

Moreover, since (B) differs from (A) only in the signs of β and γ, we shall obtain the value of $\frac{1}{r_1} + \frac{1}{\rho_1}$, if we change the signs of β and γ in (7) Hence, recollecting that this change does not alter the values of P, Q, and S, we have

$$\frac{1}{r_1} + \frac{1}{\rho_1} = \quad P\alpha - Q\beta - S\gamma \qquad \text{(8)}$$

Similarly, $$\frac{1}{r_2} + \frac{1}{\rho_2} = -P\alpha + Q\beta - S\gamma \qquad \text{(9)}$$

and, $$\frac{1}{r_3} + \frac{1}{\rho_3} = -P\alpha - Q\beta + S\gamma \qquad \text{(10)}$$

Add (7, 8, 9, 10), and we obtain the equation

$$\therefore \frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{\rho} + \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = 0,$$

which amounts to the proposition to be proved.

Want of room alone prevents us from giving Mr. Beecroft's elaborate investigation.

ON THE IMAGINARY EXPRESSIONS FOR SIN X AND COS X.

[*From a Correspondent.*]

The following method of investigating the imaginary forms for the sine and cosine of an arc has, I believe, some claim to novelty; and as it establishes those forms in their utmost generality, it may perhaps deserve to be recorded.

1. Put $e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 0$. $\qquad$ (1)

$$\therefore e^{2x\sqrt{-1}}+1=0 \therefore e^{x}=(-1)^{\frac{1}{2\sqrt{-1}}} \therefore x=\frac{\log-1}{2\sqrt{-1}},$$

that is, using Mr. Graves's general form for a logarithm,

$$x=\frac{(2k'+1)\pi\sqrt{-1}}{1+2k\pi\sqrt{-1}}\div 2\sqrt{-1}=\frac{\frac{1}{2}(2k'+1)\pi}{1+2k\pi\sqrt{-1}}.$$

Hence all the values of x which satisfy the equation (1) are comprised in this expression, and no others are comprised in it.

Again: these same values of x, and no other values, satisfy the equation $\cos(x+2k\pi x\sqrt{-1})=0$, as is evident,

$$\therefore e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}=m\cos(x+2k\pi x\sqrt{-1}),$$

where m is some constant to be determined.

Let $x=0 \therefore 1+1=m \therefore m=2 \therefore e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}=2\cos(x+2k\pi x\sqrt{-1})$.

2. The values of x which satisfy the equation

$$e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

are obviously derivable from those above, by changing $\log-1$ into $\log 1$: that is

$$x=\frac{\log 1}{2\sqrt{-1}}=\frac{k'\pi}{1+2k\pi\sqrt{-1}}.$$

And these same values of x, to the exclusion of all others, evidently satisfy the equation $\sin(x+2k\pi x\sqrt{-1})=0$

$$\therefore e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}=m\sin(x+2xk\pi\sqrt{-1})\ldots\ldots(3)$$

If to determine m we put, as before, $x=0$, we shall be led to the indefinite result $m=\frac{0}{0}$: to avoid this square each side of (3), and we shall have

$$e^{2x\sqrt{-1}}-2+e^{-2x\sqrt{-1}}=m^2\sin^2(x+2k\pi x\sqrt{-1}),$$

that is, as before proved,

$$2\cos(2x+2k\pi.2x\sqrt{-1})-2=m^2\sin^2(x+2k\pi x\sqrt{-1}).$$

Let $x=\frac{\pi}{2}$, and $k=0 \therefore -4=m^2 \therefore m=2\sqrt{-1}$. Consequently

$$\frac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{2\sqrt{-1}}=\sin(x+2k\pi x\sqrt{-1})$$

$$\frac{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}{2}=\cos(x+2k\pi x\sqrt{-1}),$$

where k is any whole number positive or negative.

We might establish these general results in a manner more strictly analogous to that by which the particular forms of Euler are usually

obtained. Thus, by the common development of e^z we have, upon su stituting first $x\sqrt{-1}$, and then $-x\sqrt{-1}$ in place of z,

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x, \quad e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x \ldots..$$

But this development, to be perfectly general, should be multiplied by 1 hence, introducing this factor, and adding,

$$e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$$

$$= \left(1^{x\sqrt{-1}} + 1^{-x\sqrt{-1}}\right)\cos x + \left(1^{x\sqrt{-1}} - 1^{-x\sqrt{-1}}\right)\sqrt{-1}\sin$$

Now it may be easily shown, either by De Moivré's theorem or by act development, that

$$1^{x\sqrt{-1}} + 1^{-x\sqrt{-1}} = 2\cos 2k\pi x\sqrt{-1},$$

$$1^{x\sqrt{-1}} - 1^{-x\sqrt{-1}} = 2(\sin 2k\pi x\sqrt{-1})\sqrt{-1}$$

$$\therefore \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} = \cos x\cos 2k\pi x\sqrt{-1} - \sin x\sin 2k\pi x\sqrt{-1}$$

$$= \cos(x + 2k\pi x\sqrt{-1}),$$

which agrees with the second of the general expressions above: an instead of adding we subtract the equations (4) we shall in like mai arrive at the first of those expressions. And this agreement may serve verify the accuracy of Mr. Graves's general logarithmic forms.

It is worthy of notice that if we were to attempt to deduce either of preceding general forms from the other by differentiation, using the comr formulas for this purpose, we should fail in that attempt: for the comr formulas for the differential of a logarithm, and of an exponential, are dedu on the hypothesis that real logarithms only are considered.

From these particular formulas the more general ones may, however, readily deduced: thus, calling the general naperian logarithm of x, log and the particular real logarithm, lx; we have

$$\log x = \frac{lx + 2k'\pi\sqrt{-1}}{1 + 2k\pi\sqrt{-1}} \quad \therefore d\log x = \frac{dlx}{1 + 2k\pi\sqrt{-1}},$$

but, by the common formula, $dlx = \dfrac{dx}{x}$ $\therefore d\log x = \dfrac{dx}{x}\cdot\dfrac{1}{1 + 2k\pi\sqrt{.}}$

Again, if $y = e^x$ $\therefore \log y = x$ $\therefore d\log y = \dfrac{dy}{y}\cdot\dfrac{1}{1+2k\pi\sqrt{-1}} = dx$

$$\therefore de^x = (1 + 2k\sqrt{-1})e^x\,dx;$$

and by employing this general formula, instead of the usual particular of it answering to $k=0$ each of the expressions referred to may be dedu from the other by differentiation.

From the foregoing equations it is easy to deduce the general forms sin x and cos x; and which may replace the imperfect forms hitherto gi when the utmost generality is desired: for put

$$x + 2kx\pi\sqrt{-1} = z \quad \therefore x = \frac{z}{1 + 2k\pi\sqrt{-1}},$$

and consequently, by substitution in the equations referred to, we have

$$\frac{e^{\frac{z\sqrt{-1}}{1+2k\pi\sqrt{-1}}}+e^{\frac{-z\sqrt{-1}}{1+2k\pi\sqrt{-1}}}}{2}=\cos z$$

$$\frac{e^{\frac{z\sqrt{-1}}{1+2k\pi\sqrt{-1}}}-e^{\frac{-z\sqrt{-1}}{1+2k\pi\sqrt{-1}}}}{2\sqrt{-1}}=\sin z.$$

Of course the imaginary $\sqrt{-1}$ need not be preserved in both numerator and denominator of the exponents, if its removal from either be desired; though, as here written, the expressions will be found sufficiently adapted to the purposes of analysis. They have the same generality, as exponential forms for $\cos z$ and $\sin z$, as Mr. Graves's forms have for logarithms.

Belfast, Dec. 5, 1844. Y.

ON POLES AND POLARS IN SPACE.

[*Mr. Fenwick.*]

In applying the polar theory to *surfaces* of the second order, in the following pages, we have merely carried out the principle employed in a former paper in connection with *lines* of the second degree. Our object in both papers (so far as the fundamental relation is concerned) has been,—to obtain a relation between the co-ordinates of two points in reference to a *line*, in the one case, and a *surface*, in the other, of the second degree, in which both co-ordinates are equally involved; then, making one of the points variable, to get finally a line in plano, and a surface in space, of the *first* degree, which have a reciprocal relation with the given point, or a relation in which the constant and variable co-ordinates are interchangeable. In accomplishing this, we trust we have made ourselves intelligible to the youngest student. It must be admitted, however, that the geometrical method (the *French* mode of treating poles and polars) carries with it, at the outset, more clearness than the co-ordinate one, especially to minds not much accustomed to the reasoning of the modern analysis. It is in the *application* that the analytical method seems to have the advantage, both in point of power and elegance. In speaking of the analytical method, we may mention, that the only work, so far as we know, in which poles and polars are treated by the co-ordinate method, is one by *Magnus*, a very distinguished German mathematician. His method, however, is very different from that employed in these papers, both as it regards the establishment of the reciprocal relation, and the application of it.

1. *To find the equation of reciprocity between a point and its polar plane, in reference to the general surface*

$$az^2+by^2+cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx+l=0\;....\;(1)$$

Definition. Two systems of points so related, that to each *point* of the one system there is a corresponding *plane* of the other system, and to each point of the second system a corresponding plane of the first system, are named a *reciprocal system*.

Or, the two systems are so connected that the co-ordinates of a given

point in the one, and the variable co-ordinates of a plane in the other, are *interchangeable* in the equation which expresses their relation.

Let, then,

$$(z-z_1)+m(y-y_1)+n(x-x_1)=0,$$

and $$(z-z_1)+m_1(y-y_1)+n_1(x-x_1)=0,$$

be the equations of two planes passing through the same point $z_1\,y_1\,x_1$. Combining these by multiplication, there results for a surface passing through the point $z_1\,y_1\,x_1$ an equation of the form

$$(z-z_1)^2+A(y-y_1)^2+B(x-x_1)^2$$
$$+C(z-z_1)(y-y_1)+D(z-z_1)(x-x_1)+E(y-y_1)(x-x_1)=0\ldots(\alpha)$$

Also,

$$(z-z_2)+p(y-y_2)+q(x-x_2)=0,$$

and $$(z-z_2)+p_1(y-y_2)+q_1(x-x_2)=0,$$

being the equations of two planes through the point $z_2\,y_2\,x_2$, the equation of a surface through this point will be of the form

$$(z-z_2)^2+A_1(y-y_2)^2+B_1(x-x_2)^2$$
$$+C_1(z-z_2)(y-y_2)+D_1(z-z_2)(x-x_2)+E_1(y-y_2)(x-x_2)=0\ldots(\beta)$$

Now, combine (α) and (β) by addition and let the result be *identical* with the surface (1); then we have the subsequent equations:

$$a(A+A_1)-2b=0\ldots\ldots(2)$$
$$a(B+B_1)-2c=0\ldots\ldots(3)$$
$$a(E+E_1)-4d=0\ldots\ldots(4)$$
$$a(D+D_1)-4e=0\ldots\ldots(5)$$
$$a(C+C_1)-4f=0\ldots\ldots(6)$$
$$a(2z_1+Cy_1+Dx_1+2z_2+C_1y_2+D_1x_2)+4g=0\ldots(7)$$
$$a(2Ay_1+Cz_1+Ex_1+2A_1y_2+C_1z_2+E_1x_2)+4h=0\ldots(8)$$
$$a(2Bx_1+Dz_1+Ey_1+2B_1x_2+D_1z_2+E_1y_2)+4k=0\ldots(9)$$
$$a(z_1^2+Ay_1^2+Bx_1^2+Ex_1y_1+Dx_1z_1+Cz_1y_1)$$
$$+a(z_2^2+A_1y_2^2+B_1x_2^2+E_1x_2y_2+D_1x_2z_2+C_1z_2y_2)-2l=0\ldots(10)$$

where $A=m\,m_1$, $B=n\,n_1$, $C=m+m_1$, $D=n+n_1$, $E=m\,n_1+m_1n$, $A_1=pp_1$, $B_1=qq_1$, $C_1=p+p_1$, $D_1=q+q_1$, and $E_1=pq_1+p_1q$.

Eliminating the *eight* arbitrary quantities m, m_1, n, n_1, *etc.*, from the *nine* equations (2, 3, 4, 5, 6, 7, 8, 9, 10) we get the relation

$$(az_2+fy_2+ex_2+g)z_1+(by_2+fz_2+dx_2+h)y_1$$
$$+(cx_2+ez_2+dy_2+k)x_1+gz_2+hy_2+kx_2+l=0,$$

or, $$(az_1+fy_1+ex_1+g)z_2+(by_1+fz_1+dx_1+h)y_2$$
$$+(cx_1+ez_1+dy_1+k)x_2+gz_1+hy_1+kx_1+l=0.$$

Hence, if $z_2\,y_2\,x_2$ be the given point, the polar plane of this point ($z\,y\,x$ being written for $z_1\,y_1\,x_1$) will be expressed by the equation

$$(az_2+fy_2+ex_2+g)z+(by_2+fz_2+dx_2+h)y$$
$$+(cx_2+ez_2+dy_2+k)x+gz_2+hy_2+kx_2+l=0\ldots(11)$$

And if $z_1\,y_1\,x_1$ be the given point, its polar plane will be

$$(az_1+fy_1+ex_1+g)\,z+(by_1+fz_1+dx_1+h)y$$
$$+(cx_1+ez_1+dy_1+k)x+gz_1+hy_1+kx_1+l=0\ldots(12)$$

If instead of (1) we had employed the surface

$$az^2 + by^2 + cx^2 + 2ex = 0 \text{...........................} (13)$$

the polar plane of a point $z_1\ y_1\ x_1$ would have been

$$az_1z + by_1y + (cx_1 + e)\,x + ex_1 = 0 \text{..............} (14)$$

2. *Co-ordinates of the pole.*

The co-ordinates $z_1\ y_1\ x_1$ of the pole of a given polar plane can be found, by considering this polar plane and (12) *identical*, and equating the coefficients of the same powers of the variables.

3. *The poles of three or more planes which pass through the same point, lie in one plane, the polar plane of this point; and, conversely, the polar planes of three or more points which lie in one plane, pass through the same point, the pole of this plane.*

Let $z_1\ y_1\ x_1$, $z_2\ y_2\ x_2$ and $z_3\ y_3\ x_3$ be the co-ordinates of the poles of three planes which intersect in the point $v_1\ u_1\ t_1$: then by (14) *article* 1, the planes will be represented by the equations

$$az_1z + by_1y + (cx_1 + e)\,x + ex_1 = 0 \text{..............} (15)$$

$$az_2z + by_2y + (cx_2 + e)\,x + ex_2 = 0 \text{..............} (16)$$

$$az_3z + by_3y + (cx_2 + e)\,x + ex_3 = 0 \text{..............} (17)$$

and the polar of the point $v_1\ u_1\ t_1$ by the equation

$$av_1z + bu_1y + (ct_1 + e)\,x + et_1 = 0 \text{..............} (18)$$

Since the planes (15, 16, 17) intersect in the point $v_1\ u_1\ t_1$, they must be satisfied by the co-ordinates $v_1\ u_1\ t_1$; hence

$$az_1v_1 + by_1u_1 + (cx_1 + e)\,t_1 + ex_1 = 0,$$
$$az_2v_1 + by_2u_1 + (cx_2 + e)\,t_1 + ex_2 = 0,$$
$$az_3v_1 + by_3u_1 + (cx_3 + e)\,t_1 + ex_3 = 0;$$

or, arranging according to $z_1y_1x_1$, *etc.*,

$$av_1z_1 + bu_1y_1 + (ct_1 + e)\,x_1 + et_1 = 0,$$
$$av_1z_2 + bu_1y_2 + (ct_1 + e)\,x_2 + et_1 = 0,$$
$$av_1z_3 + bu_1y_3 + (ct_1 + e)\,x_3 + et_1 = 0.$$

Now the last equations are what (18) becomes by substituting successively for $z\ y\ x$ the values $z_1y_1\ x_1$, $z_2\ y_2\ x_2$ and $z_3\ y_3\ x_3$, and, consequently, the equation (18) or the polar of the point $v_1\ u_1\ t_1$, is satisfied by these co-ordinates. It follows, then, that the poles of the three planes which intersect in the same point, lie in the polar of this point. The converse is proved in a similar way.

Similar reasoning will also apply to any number of planes; hence the following general theorem:

If a plane turn about a point in itself, its pole will move upon a plane, the polar plane of that point, and, conversely, if a point move upon a plane, its polar plane will turn round a point, the pole of that plane.

4. *Reciprocal straight lines.*

If a plane turn round a fixed straight line, its pole will move in a straight line, and conversely, if a point move in a straight line, its polar will turn round a fixed straight line.

For when a plane turns round a fixed straight line it is the same as if it turned round *two fixed points* in that line, and, therefore, its pole (*art.* 3.)

moves in two planes, the polars of these points, that is, it moves in the straight line in which the two planes meet.

Definition. Two lines related as in this article are named reciprocal straight lines.

5. *If three or more straight lines intersect in a point, their reciprocal straight lines lie in a plane; and, conversely, if three or more straight lines lie in a plane, their reciprocal straight lines pass through the same point.*

For let planes revolve about the following sets of points, taken two and two, *viz*, $z\ y\ x$, $z_1\ y_1\ x_1$; $z\ y\ x$, $z_2\ y_2\ x_2$; $z\ y\ x$, $z_3\ y_3\ x_3$; *etc*, of which the point $z\ y\ x$ is common to each plane: then (*art.* 4) there will be one plane, the polar of $z\ y\ x$, common to the planes, which, by their intersection, form the lines that are reciprocal to the lines passing through the points $z\ y\ x$, $z_1\ y_1\ x_1$; $z\ y\ x$, $z_2\ y_2\ x_2$; *etc*, and, consequently, all these reciprocal straight lines will be situated in the *common* plane.

6. *To find the equation of reciprocity between two polar lines.*

Let

$$\left.\begin{array}{l} y = \alpha z + \beta \\ x = \alpha' z + \beta' \end{array}\right\} \dots\dots\dots\dots (a)$$

be the equations of a line of which it is proposed to find the reciprocal straight line. The reciprocal of (a) is (*art.* 4) the intersection of two planes which are the polars of any two points in (a). If, then, we find the polar of *any* point $z\ y\ x$ in (a) and consider $z\ y\ x$ in the result to have *all possible values*, it will be evident, that any planes resulting therefrom will, by their intersection, form the line which is the reciprocal of (a). Whence if from (a) we insert the values of y and x in (14) article 1, we shall have after arranging for z,

$$z(az_1 + \alpha b y_1 + \alpha' c x_1 + \alpha' e) + \beta b y_1 + \beta' c x_1 + \beta' e + e x_1 = 0,$$

or, from the *indeterminateness* of z and suppression of accents,

$$\left.\begin{array}{l} az + \alpha b y + \alpha' c x + \alpha' e = 0 \\ \text{and} \quad \beta b y + \beta' c x + \beta' e + e x = 0 \end{array}\right\} \dots\dots\dots\dots (19)$$

Hence (19) are the equations of two planes which form the line required; the equation, therefore, of reciprocity is completely established.

7. *The polar of a line which moves parallel to a given line is always situated in a given plane.*

For in the equations (a) of *last article*, let α and α' be constant, and β and β' variable, these will then be the equations of a line moving parallel to a given line.

Now, the polar of this line, as we have shewn, is the intersection of the planes (19) the former of which, since α and α' are constant, is a given plane; hence the enunciated theorem.

Cor. The polar of the line which generates a cylindrical surface is always in the same plane.

8. *Diameters.*

The poles of all the planes which are parallel to one and the same plane lie in the same straight line.

Definition. The straight line which contains the poles of all the planes that are parallel to a given plane is named the diameter of the system which

is conjugate to this plane. The diameter and the given plane are sometimes said to be *conjugate* to one another.

To establish the theorem enunciated above, let

$$z = my + nx + p \text{ ....(20)}$$

be the equation of a plane in which m and n are *constant* and p *variable*; this equation then will represent all planes that are parallel to a given plane. In order to find the locus of the poles of these parallel planes (*art.* 2) let (20) be *identical* with (12) *art.* 1; then we have

$$by_1+fz_1+dx_1+h+m(az_1+fy_1+ex_1+g) = 0,$$
$$cx_1+ez_1+dy_1+k+n\,(az_1+fy_1+ex_1+g) = 0,$$
$$gz_1+hy_1+kx_1+l+p\,(az_1+fy_1+ex_1+g) = 0;$$

or, arranging for $z_1\ y_1\ x_1$ and suppressing accents:

$$fz+by+dx+h+m(az+fy+ex+g) = 0 \text{ ....(21)}$$
$$ez+dy+cx+k+n\,(az+fy+ex+g) = 0 \text{ ....(22)}$$
$$gz+hy+kx+l+p\,(az+fy+ex+g) = 0 \text{ ....(23)}$$

Now since m and n are *constant*, (21, 22) are the equations of a *given straight line*, the diameter of that system which is *conjugate* to the plane (20).

9. *Centre and diametral plane.*

The diameters of a system either pass through the same point, are parallel to one another, or pass through several indeterminate points.

Definition. The point through which the diameters pass is named the *centre* of the system, and the plane which contains the centre is named the *diametral plane* of this system.

For in the equations (21, 22) of a diameter, if we suppose m and n to be *arbitrary*, these equations will then represent all diameters in the system (1) *article* 1; hence we have from the *indeterminateness* of m and n,

$$fz+by+dx+h = 0,$$
$$az+fy+ex+g = 0,$$
$$ez+dy+cx+k = 0.$$

The values of $z\ y\ x$ (*corresponding to a centre*) deduced from these equations which have a common denominator

$$f(abc-ad^2-be^2-cf^2+2def),$$

are consequently real and finite, unless

$$abc-ad^2-be^2-cf^2+2def = 0;$$

in such case, the co-ordinates of the centre are either ∞ or $\frac{0}{0}$, that is, the centre is either at an infinite distance, or there are several indeterminate centres.

10. *To find the equation of a tangent plane.*

Definition. A diameter and a tangent plane passing through the same point are said to be *conjugate* to one another.

Let $z_1y_1x_1$ be the point of contact of the tangent plane with the surface

$$az^2+by^2+cx^2+2ex = 0;$$

then the tangent plane in general will be denoted by the equation

$$z-z_1+p\,(y-y_1)+q\,(x-x_1) = 0 \text{ ....(24)}$$

Now if we consider a diameter to pass through the point $z_1y_1x_1$, it will follow from the definition, that (24) will be parallel to the plane which is the reciprocal of any point in this diameter, and hence it will be parallel to the plane which is the polar of $z_1y_1x_1$. Consequently by (14) *art.* 1,

$$p=\frac{by_1}{az_1};\ q=\frac{cx_1+e}{az_1};$$

and, therefore, (24) becomes

$$az_1z+by_1y+(cx_1+e)x-az^2_1-by^2_1-cx^2_1-ex_1=0,$$

or, $$az_1z+by_1y+(cx_1+e)x+ex_1=az^2_1+by^2_1+cx^2_1+2ex_1.$$

But since $z_1y_1x_1$ is a point in the surface,

$$az^2_1+by^2_1+cx^2_1+2ex_1=0,$$

whence the equation of the tangent plane is

$$az_1z+by_1y+(cx_1+e)x+ex_1=0 \dots\dots\dots\dots (25^*)$$

which is the same as the polar plane of the point $z_1y_1x_1$.

From this we deduce the following

Cor. The tangent plane at any point $z_1y_1x_1$ of a curve surface can be regarded as the polar of this point; and, conversely, this point can be regarded as the pole of the tangent plane at the same point.

Had the surface of reference been (1) *art.* 1, the tangent plane at a point $z_1y_1x_1$ would have been

$$(az_1+fy_1+ex_1+g)z+(by_1+fz_1+dx_1+h)y$$
$$+(cx_1+ez_1+dy_1+k)x+gz_1+hy_1+kx_1+l=0 \,..(26)$$

11. *The polar in respect of a point without a surface of the second degree is the plane of contact of three or more tangent planes drawn to the surface from that point.*

For by the *last article*, the tangent planes are the polars of the several points of contact, and by *article* 3, these points of contact are in the same plane, the polar of the point through which the tangent planes pass.

Cor. 1. *If from a point without a surface of the second degree any number of tangent planes be drawn to the surface the several points of contact will be situated in the same plane.*

Cor. 2. *If the angular point of a solid angle move upon a plane, whilst the planes containing the angle touch a surface of the second degree, whereby the angle is in general varied, the planes of contact will turn about the same point, the pole of the plane upon which the angular point moves; and, conversely, if the plane of contact of a surface of the second degree turn about a fixed point, the tangent planes drawn from the points of contact will intersect in the same plane, the polar of the fixed point.*

12. *The line which joins the points of contact of two tangent planes to a surface of the second degree and the line of intersection of the two planes are reciprocal lines.*

This follows at once from *article* 4, in which it is shewn, that the line formed by the intersection of two planes is reciprocal to the line which passes through the poles of these planes, that is, in this case, the line which joins the points of contact (*art.* 10, *cor.*).

* This, in connection with (14) *art.* 1, contains a solution of exercise 29, No. 4, of the Mathematician.

Cor. If from two points without a surface of the second degree tangent planes be drawn to the surface, then the line formed by the intersection of the two planes of contact is reciprocal to the line joining the two points.

13. *The reciprocals of three or more straight lines situated in the same plane without a surface of the second degree are chords of the surface passing through the same point.*

This is readily deduced from *articles* 5 and 12.

14. *If in a surface of the second degree a polyhedron be inscribed and through its angular points tangent planes be drawn so that a polyhedron may circumscribe the same surface: then, the faces of the one will be polars of the angular points of the other: Also, the lines formed by the intersecting planes of the one will be reciprocal to the lines formed by the intersecting planes of the other.*

The demonstration of the above theorem is contained in *articles* 10, 11, and 12.

15. *An indefinite number of surfaces of the second order may pass through eight given points in space. Let sections be cut parallel to a given plane: then will the diameters which are conjugate to these sections all meet in the same point.*

The general equation of a surface of the second order may be written thus,

$$az^2+by^2+cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx+1=0,$$

so as to contain *nine* unknown coefficients. To determine these coefficients, in the present case, there will be given *eight* equations only, corresponding to the eight given points.

Hence one of the coefficients will remain *indeterminate*, and, therefore, an indefinite number of such surfaces will pass through the eight points; this proves the first part of the theorem.

Again,

$$z=my+nx+p,$$

being the equation of a plane parallel to a given plane, the equations of a diameter conjugate to this plane (*art.* 8) are

$$fz+by+dx+h+m(az+fy+ex+g)=0,$$
$$ez+dy+cx+k+n(az+fy+ex+g)=0,$$

of which, one of the quantities a, b, c, *etc.*, is *indeterminate.*

Let a be the arbitrary; then we have

$$z=0,$$
$$(b+mf)y+(d+me)x+h+mg=0,$$
$$(d+nf)y+(c+ne)x+k+ng=0;$$

the diameters then, in this case, all pass through a point in the plane of xy; the co-ordinates of which (that is, the values of x and y in these equations) are

$$y=\frac{hc-dk+g(mc-dn)+e(hn-km)}{d^2-bc+e(dm-bn)+f(dn-mc)},$$
$$x=\frac{bk-hd+g(bn-md)+f(km-nh}{d^2-bc+e(dm-bn)+f(dn-mc)}.$$

We might shew in a similar way, that if b be the arbitrary, the point through which the diameters pass is in the plane of xz; and that if c be arbitrary, it is in the plane of yz. Also, d, e, f being respectively, the indeterminate quantities, the diameters in the first case meet in the axis of z, in the second, in the axis of y, and in the third, in the axis of x.

Next, let g be the arbitrary, then

$$fz+by+dx+h=0,$$
$$ez+dy+cx+k=0.$$

The diameters, therefore, pass through several indeterminate points: and the same is the case when h or k is the arbitrary quantity.

Scholium. From the preceding investigation the following would appear to be the enunciation of the theorem in general of this article:

An indefinite number of surfaces of the second order may pass through eight given points in space. Let sections be cut parallel to a given plane; then will the diameters which are conjugate to these sections either pass through the same point, or several indeterminate points.

16. *If from a given point tangent planes be drawn to all surfaces of the second degree which pass through eight given points in space, then will the planes of contact intersect in one and the same straight line.*

Denote the surfaces in general by the equation

$$az^2+by^2+cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx+1=0.$$

Then $z_1 y_1 x_1$ being the given point, the planes of contact (*art.* 11) will be denoted by the equation

$$(az_1+fy_1+ex_1+g)z+(by_1+fz_1+dx_1+h)y$$
$$+(cx_1+ez_1+dy_1+k)x+gz_1+hy_1+kx_1+1=0.$$

Now since the surfaces pass through *eight* given points only, one of the quantities, a, b, c, *etc.*, will be *indeterminate*. Let a be that quantity, then $z=0$,

$$\therefore (by_1+fz_1+dx_1+h)y+(cx_1+ez_1+dy_1+k)x+gz_1+hy_1+kx_1+1=0,$$

the equation of a line in the plane of xy, in which all the planes of contact intersect.

17. *The planes of contact of one and the same point in reference to all surfaces of the second degree which pass through seven given points in space intersect in the same point.*

In this case (denoting the surfaces and planes of contact as in last article), since the surfaces pass through seven given points only, *two* of the quantities a, b, c, *etc.*, will be *indeterminate*. Let these be a and b; then

$$z=0, y=0,$$
$$(cx_1+ez_1+dy_1+k)x+gz_1+hy_1+kx_1+1=0$$

Hence the planes of contact will pass through the same point in the axis of x.

(To be continued.)

MODERN GEOMETRY.

(*Continued from page* 181.)

XI.

There is also another set of equations of three terms, each equation of which contains eight segments, and which are therefore different from the preceding ones.

$$(1) \ldots \frac{ac.ac'}{ab.ab'} + \frac{bc.bc'}{ba.ba'} = 1 \qquad (1') \ldots \frac{ac.ac'}{ab.ab'} + \frac{b'c.b'c'}{b'a.b'a'} = 1$$

$$(2) \ldots \frac{ab.ab'}{ac.ac'} + \frac{cb.cb'}{ca.ca'} = 1 \qquad (2') \ldots \frac{ab.ab'}{ac.ac'} + \frac{c'b'.c'a'}{c'a.c'a'} = 1$$

$$(3) \ldots \frac{a'c.a'c'}{a'b.a'b'} + \frac{bc.bc'}{ba.ba'} = 1 \qquad (3') \ldots \frac{a'c.a'c'}{a'b.a'b'} + \frac{b'c.b'c'}{b'a.ba'} = 1$$

$$(4) \ldots \frac{a'b.a'b'}{a'c.a'c'} + \frac{cb.cb'}{ca.ca'} = 1. \qquad (4') \ldots \frac{a'b.a'b'}{a'c.a'c'} + \frac{c'b.c'b'}{c'a.c'a'} = 1.$$

1. For by (VIII. 1) we have, supposing m to coincide with a,

$$\frac{ac.ac'}{ab.ab'} = \frac{a\gamma}{a\beta};$$

and suppose, again, that m coincides with b, we have

$$\frac{bc.bc'}{ba.ba'} = -\frac{\beta\gamma}{\beta a}.$$

Whence, adding, and bearing in mind that $a\beta = \gamma a - \beta\gamma$,

$$\frac{ac.ac'}{ab.ab'} + \frac{bc.bc'}{ba.ba'} = \frac{a\gamma - \beta\gamma}{a\beta} = 1;$$

which is the first of the four equations on the left side of the vertical line.

The remaining three may be similarly found: or they may be written out at once, on the principle of symmetry.

2. The four corresponding equations are found from these by the substitution of equal anharmonic ratios, and need not be further dwelt upon.

XII.

The equation of involution by means of six segments may also be put under the twelve following forms, any one of which implies the eleven remaining ones.

1st. involving the segment aa'.

$$(1) \ldots\ldots \frac{ab.a'c}{aa'.bc} + \frac{ab'.a'c'}{aa'.b'c'} = 1; \text{ see below.}$$

$$(2) \ldots\ldots \frac{ab.a'c'}{aa'.bc'} + \frac{ab'.a'c}{aa'.b'c} = 1; \text{ change } c,\ c' \text{ in } (1)$$

$$(3) \ldots\ldots \frac{ac.a'b}{aa'.cb} + \frac{ac'.a'b'}{aa'.c'b'} = 1; \text{ change } b,\ c, \text{ and also } b',c' \text{ in } (1)$$

$$(4) \ldots\ldots \frac{ac.a'b'}{aa'.cb'} + \frac{ac'.a'b}{aa'.c'b} = 1; \text{ change } b,\ b' \text{ in } (3)$$

2nd. involving the segment bb'.

(5) .... $\dfrac{bc.b'a}{bb'.ca} + \dfrac{bc'.b'a'}{bb'.c'a'} = 1$; circulating the letters one step from (1)

(6) .... $\dfrac{bc.b'a'}{bb'.ca'} + \dfrac{bc'.b'a}{bb'.c'a} = 1$; ditto from (2)

(7) .... $\dfrac{ba.b'c}{bb'.ac} + \dfrac{ba'.b'c'}{bb'.a'c'} = 1$; ditto from (3)

(8) .... $\dfrac{ba.b'c'}{bb'.ac'} + \dfrac{ba'.b'c}{bb'.a'c} = 1$; ditto from (4)

3rd. involving the segment cc'.

(9) .... $\dfrac{ca.c'b}{cc'.ab} + \dfrac{ca'.c'b'}{cc'.a'b'} = 1$; circulating the letters one step from (5)

(10).... $\dfrac{ca.c'b'}{cc'.ab'} + \dfrac{ca'.c'b}{cc'.a'b} = 1$; ditto from (6)

(11).... $\dfrac{cb.c'a}{cc'.ba} + \dfrac{cb'.c'a'}{cc'.b'a'} = 1$; ditto from (7)

(12).... $\dfrac{cb.c'a'}{cc'.ba'} + \dfrac{cb'.c'a}{cc'.b'a} = 1$; ditto from (8).

Any further circulation brings us to a successive repetition of the steps already passed over; and hence this table comprehends the entire series of this class of properties.

The first, or fundamental equation is thus obtained:

One of the forms of the expression of the anharmonic ratio, analogously formed to those in (V.), is

$$\frac{ab}{ad} : \frac{cb}{cd} + \frac{a'c'}{a'd'} : \frac{b'c'}{b'd'} = 1.$$

Now in attending to the notation used in (V) and that here employed it will be seen that the two letters in vertical lines in the following arrangement correspond to each other.

$abcd$	$a'b'c'd'$
$abca'$	$a'b'c'a$

Making the requisite change, viz. putting a' for d and a for d', the above expression becomes the first of our eighteen in this proposition.

Also since the several points are symmetrically involved in pairs in an involution, the permutations which we have made are justified on the general principles of symmetry. They may, however, be each deduced independently.

XIII.

In an involution of eight points, the anharmonic ratio of any four is equal to the anharmonic ratio of the remaining four.

For by the second Apollonian property (IX) we have, since $abc\ a'b'c'$, and $abd\ a'b'd'$ form involution systems,

$$\frac{ac}{bc} = \frac{a'c'}{b'c'} \cdot \frac{ab'}{a'b}, \text{ and } \frac{bd}{ad} = \frac{b'd'}{a'd'} \cdot \frac{a'b}{ab'};$$

and compounding these, we get

$$\frac{ac.bd}{bc.ad} = \frac{a'c'}{b'c'}.\frac{b'd'}{a'd'}, \text{ or}$$

$$\frac{ac}{bc}:\frac{ad}{bd} = \frac{a'c'}{b'c'}:\frac{a'd'}{b'd'}.$$

In the same way may the other parts implied in the general statement of the proposition be established.

Scholium. When five, six, or more pairs of conjugate points in involution are given, corresponding properties may be deduced. In the application that will here be made of this principle, we shall not, however, require them; and hence it will not be necessary to enter further into the subject under this aspect.

XIV.

We may now proceed to the consideration of some special cases of the general principle.

1. If one of the points c or c' be at o, the centre of involution, the other will be infinitely distant; and conversely, if one of the points c' or c be infinitely distant, the other will coincide with the centre of involution.

For since $ao.oa' = co.oc'$, this can only be fulfilled on the conditions alleged in the enunciation.

Cor. The seven equations of involution become on this hypothesis:—

$$ao.oa' = bo.ob';$$

$$\frac{ab.ba'}{ab'.b'a'} = \frac{bo}{b'o} \quad \Big| \quad \frac{b'a.ab}{ba'.a'b'} = \frac{ao}{a'o};$$

$$\frac{ab'}{a'b} = \frac{ob'}{oa'} \quad \Big| \quad \frac{ab}{a'b} = \frac{ob}{oa'}$$

$$\frac{ab'}{a'b} = \frac{oa}{ob} \quad \Big| \quad \frac{ab}{ab'} = \frac{oa}{ob'}$$

and any one of these implies the six others.

2. Let a point of the system a, b, c, coincide with a point of the system $a'b'c'$; and let c,c' be those points, and coalescing in e. Then the equations of Apollonius, and those of Desargues become respectively

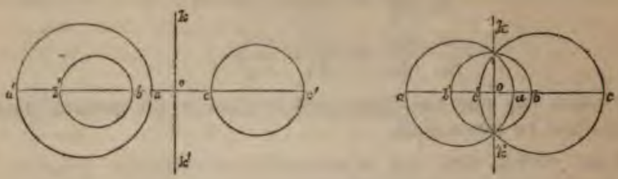
Apollonius :—	Desargues :—
$\dfrac{ae.eb}{a'e.eb'} = \dfrac{ab}{a'b'}$	$\dfrac{ab.ab'}{a'b'.a'b} = \dfrac{ae^2}{a'e^2}$
$\dfrac{ae.eb'}{a'e.eb} = \dfrac{ab'}{a'b}.$	$\dfrac{ba.ab'}{b'a'.a'b} = \dfrac{be^2}{b'e^2}.$

Desargues has treated of this case under the title of *involution of five points:* and Chasles has properly named the point e a *double point.*

Chasles deduces from this consideration the property which we have here made our definition of involution.

The definition of involution, as adopted in this paper, or the property deduced in the preceding case after Chasles's method from his own defini-

tion, connects itself in a remarkable manner with a known property of a system of circles, viz. those of the *radical axis*.



Any two circles being given on a plane or on the sphere, it is well known that a line exists from any point of which secants (or chords as the case may be) being drawn to the circles, the rectangles under the segments of these secants (or chords) intercepted by the circles and estimated from the point in the said line, will be equal to one another.

This property has been long known to English mathematicians, as far as regards the circle in plano; but the important consequences of it have only been developed by Foreign writers—as Gaultier, Steiner, Durrande, Hachette, Cauchy, Français and Monge. Some notes on the subject in reference to spheres may be seen in *Leybourn's Repository, vol. v.*; and in reference to circles on the sphere, in *Hutton vol. ii.* We may therefore assume the leading properties as being known.

When the two circles are either each without the other, or one within the other, the radical axis kk' is without them both; but when the one circle cuts the other, the radical axis is the chord joining their intersections; and when they touch each other, it is their common tangent.

Now let the line joining the centres of two circles meet those circles in a,a' and b,b' and let kk' be the radical axis, cutting it in o; then $ao.oa' = bo.ob'$: and if any number of pairs of points be taken in the same line, viz. cc', dd', *etc*, subject to the same conditions, and circles be described upon these as diameters, kk' will be the radical axis of each pair of circles so described; or which comes to the same thing kk' will be the common radical axis of all the circles, and o will be their *radical centre*.

Hence we obtain this theorem :—

If any number of circles have the same radical axis, the points of intersection of these with the line which contains their centres will be divided in involution, and will possess all the properties deduced in the former part of this discussion.

Also :—

If through o the radical centre of such a system of circles, any line be drawn to cut the circles in other points $\alpha, \beta, \gamma, \ldots$ and $\alpha', \beta', \gamma', \ldots$, then this line will also be divided in involution, and o will be the centre of such involution.

And :—

If such line should touch one of the circles in ϵ and cut the others, then ϵ will be a double point.

4. It will be seen at once, that if a and a' be both on one side of o, then b and b' must also be on one side of o, c and c' on one side of o, *etc.* When a and a' are on different sides of o, b and b' must be so too, as well as c and c'

etc: but it is not necessary that a, b, c, *etc.* should all be on the same side of o.

Moreover, when a and a' are on the same side of o, then all the circles when b, c, *etc.*, are on that side, will fall successively each within the preceding: but when a and a' are on different sides of o, then all the circles will intersect in two points, k and k', in the radical axis equidistant from o.

5. When, in the first figure, any one of the circles is reduced to a point, by the coincidence of (suppose) d and d', which call ω, then ω is a ***double point***, or that which we have called the ***focus of involution.*** It is obvious, too, that another point ω' on the opposite side exists and is ***always real.*** In the second figure, these points are situated at the intersections of the circle on kk' with the common diameter of the circles; and hence can never coalesce.

Scholium. This connection between the circle and the system of involution will lead to important results—the insertion of which, however, must for the present be deferred.

Chasles nowhere refers to this connection, though he might possibly have noticed it. Still had he done so, it is likely that it might have led him to discover that in the second figure, the foci of involution (viewed as double points) were imaginary; and that more simply than by analysis, or than by the abstract consideration of his system. *Apercu, p.* 313.

XV.

It only remains, in completion of our objects in this part of the discussion to notice a few general consequences of the preceding results.

1. If we deem lines Sa, Sb, Sc, and Sa' Sb', Sc', *etc.* from the points of involution to any point whatever S, and denote these lines by A, B, C, A', B', C', *etc.*: then will the same relation as to involution hold with respect to the sines of the angles formed by these lines, that holds with respect to the segments of the line $abc\,a'b'c'$, *etc.*

For in (X.) we have by means of (II. 2)

$$\frac{ab}{ac}:\frac{a'b}{a'c}=\frac{\sin AB}{\sin AC}:\frac{\sin A'B}{\sin A'C}, \text{ and}$$

$$\frac{a'b'}{a'c'}:\frac{ab'}{ac'}=\frac{\sin A'B'}{\sin A'C'}:\frac{\sin AB'}{\sin AC'}: \text{ whence}$$

$$\frac{\sin AB}{\sin AC}:\frac{\sin A'B}{\sin A'C}=\frac{\sin A'B'}{\sin A'C'}:\frac{\sin AB'}{\sin AC'}.$$

And it follows obviously that all the other formulæ will be capable, by a mere change in the mode of writing, of conversion into corresponding true formulæ. Such a system of lines we shall call a ***system of involution-radiants.***

2. If a ***system of involution radiants*** be cut by any transversal whatever, its segments will be in involution.

This is the converse of the preceding and flows from it at once.

3. If a system of planes all meeting in one line be substituted for the system of linear radiants in (1) and (2), the same will hold true, and the proof is formed in the same manner by means of (II. 3).

4. Every projection of an involution system by means of linear radiants

upon another line in the same plane with the original one and with the projecting point, is also an involution system.

This is an obvious consequence of (1) and (2).

5. Every projection of an involution system upon any line whatever by means of planes which all meet in one line, will also form an involution system.

For the planes themselves form an involution of angles, and hence the properties of anharmonic ratio exist amongst them as before: and the line upon which the projection is made cutting these, is, also, divided in involution.

6. The same property may, in nearly the same way, be proved to hold good upon the sphere.

It will, however, in analogy to our definition of an involution-system, be possible to express it on the sphere by

$$\tan \tfrac{1}{2} oa \tan \tfrac{1}{2} oa' = \tan \tfrac{1}{2} ob \tan \tfrac{1}{2} ob' = \tan \tfrac{1}{2} oc \tan \tfrac{1}{2} oc';$$

and then by a process very similar to that in plano, we shall obtain

$$\frac{\sin ab}{\sin ac} : \frac{\sin a'b}{\sin a'c} = \frac{\sin a'b'}{\sin a'c'} : \frac{\sin ab'}{\sin ac'};$$

and the rest follows of course.

7. The centre of involution is not, generally, projective: that is, the projection of the centre of involution is not the centre of the projected involution, except in particular cases.

XVI.

When three straight lines radiating from the same point s, cut the circumference of a circle in a and a', b and b', c and c': there will exist amongst the segments of the circle the following relations, having a striking analogy to the expression of involution:

$$\frac{\sin \frac{1}{2} ca \sin \frac{1}{2} ca'}{\sin \frac{1}{2} cb \sin \frac{1}{2} cb'} = \frac{\sin \frac{1}{2} c'a \sin \frac{1}{2} c'a'}{\sin \frac{1}{2} c'b \sin \frac{1}{2} c'b'},$$

$$\frac{\sin \frac{1}{2} ab \sin \frac{1}{2} ab'}{\sin \frac{1}{2} ac \sin \frac{1}{2} ac'} = \frac{\sin \frac{1}{2} a'b \sin \frac{1}{2} a'b'}{\sin \frac{1}{2} a'c \sin \frac{1}{2} a'c'},$$

$$\frac{\sin \frac{1}{2} bc \sin \frac{1}{2} bc'}{\sin \frac{1}{2} ba \sin \frac{1}{2} ba'} = \frac{\sin \frac{1}{2} b'c \sin \frac{1}{2} b'c'}{\sin \frac{1}{2} b'a \sin \frac{1}{2} b'a'}.$$

The reader, supplying the figure as described, and drawing ac', $a'c$, ab', $a'b$, bc', $b'c$, ca', ab', will find a series of similar triangles, from which the conclusion is readily obtained. We shall give the process for the first equation.

$$\begin{array}{llll} as : sc' & :: ac : a'c', & \text{from} & asc',\ a'sc', \\ sc' : sb & :: b'c' : bc & \ldots. & bsc',\ b'sc', \\ sc : sa & :: a'c : ac' & \ldots. & asc',\ a'sc', \\ sb : sc & :: bc' : cb' & \ldots. & bsc',\ b'sc', \end{array}$$

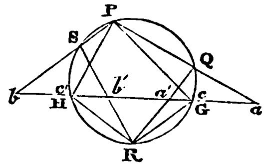
whence compounding, and forming the fraction, we get in *lines*

$$\frac{ca.ca'}{cb.cb'} = \frac{c'a.c'a'}{c'b.c'b'}.$$

Then proceeding as in (II. 5), we get the property enunciated.

Scholium. It is obvious that all the properties deduced as belonging to the system of involution, will apply to the system of chords, or of the sines of semi-arcs belonging to the present mode of division.

XVII.



1. Let PQRS be a quadrilateral inscribed in a circle, and GH any transversal cutting the circle in G and H; join PG, PH, RG, RH, P and R being either pair of opposite angular points: then will the anharmonic ratios of the angles at P and R be equal.

Denote PQ by A, PG by B, PS by C, and PH by D; and the lines from R to the same points by A′, B′, C′, D′: then

$$\frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD} = \frac{\sin A'C'}{\sin A'D'} : \frac{\sin B'C'}{\sin B'D'},$$

and the other two corresponding equations.

For by the circle we have

$$\sin AC = \sin A'C' \qquad \sin BC = \sin B'C'$$
$$\sin AD = \sin A'D' \qquad \sin BD = \sin B'D',$$

which justifies the proposition.

2. The transversal is divided, at its points of intersection with the circle and quadrilateral, in involution.

Let A, A′ meet the transversal in a, a'; D, D′ in b, b'; and let the intersections of it by the circle, viz. G and H, be c, c'. Then we have

$$\frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD} = \frac{ac'}{ab} : \frac{cc'}{cb},$$

$$\frac{\sin A'C'}{\sin A'D'} : \frac{\sin B'C'}{\sin B'D'} = \frac{a'c'}{a'b'} : \frac{cc'}{cb'} :-$$

Whence by (1) we have at once

$$\frac{ac'}{ab} : \frac{cc'}{cb} = \frac{a'c'}{a'b'} : \frac{cc'}{cb'},$$

$$\text{or,} \qquad a'b'.bc.c'a = ab.b'c.c'a'.$$

This (IX) is the 1st criterion of involution of Pappus.

3. The same properties that have just been proved with respect to the circle characterise every conic section.

For every conic section may be so projected * as to become a circle. Let then $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be the original points of section of the transversal with a conic and its inscribed quadrilateral; and $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ the lines corresponding to A, A′, B, B′, C, C′, D, D′ of the projected figure. Also let abc, $a'b'c'$, correspond to $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$.

* In a future part of this series of papers it is intended to give a chapter on the principles and application of the method of projections: for the present, as the properties we shall now have occasion to employ are very simple and familiarly known, they may be assumed to be true.

Then (xv. 5) the anharmonic ratios will be

$$\frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD} = \frac{\sin A_1C_1}{\sin A_1D_1} : \frac{\sin B_1C_1}{\sin B_1D_1},$$

$$\frac{\sin A'C'}{\sin A'D'} : \frac{\sin B'C'}{\sin B'D'} = \frac{\sin A_2C_2}{\sin A_2D_2} : \frac{\sin B_2C_2}{\sin B_2D_2} :—$$

whence by (1) we have

$$\frac{\sin A_1C_1}{\sin A_1D_1} : \frac{\sin B_1C_1}{\sin B_1D_1} = \frac{\sin A_2C_2}{\sin A_2D_2} : \frac{\sin B_2C_2}{\sin B_2D_2} :$$

which proves the first part of the proposition.

Again, we have

$$\frac{\sin A_1C_1}{\sin A_1D_1} : \frac{\sin B_1C_1}{\sin B_1D_1} = \frac{\alpha\gamma'}{\alpha\beta} : \frac{\gamma\gamma'}{\gamma\beta},$$

$$\frac{\sin A_2C_2}{\sin A_2D_2} : \frac{\sin B_2C_2}{\sin B_2D_2} = \frac{\alpha'\gamma'}{\alpha'\beta'} : \frac{\gamma\gamma'}{\gamma\beta'} :—$$

whence will follow, as in the former case, the criterion of involution,

$$\alpha'\beta'.\beta\gamma.\gamma'\alpha = \alpha\beta.\beta'\gamma.\gamma'\alpha'.$$

4. *Conversely:*—if we have two radial systems, each of four line which correspond one to one, and if the anharmonic ratios of these be then the radiants of one system intersect the corresponding ones other, in points which with the two centres of radiation are upon the conic section.

For a conic section may be described through any five of these six p let one be drawn through the two centres P, R and three of the sections Q, G, H. Then, if it do not also pass through S, let it cut some other point S′, and join PS′, which call Δ: and by the direct sition we shall have

$$\frac{\sin A'C'}{\sin A'D'} : \frac{\sin B'C'}{\sin B'D'} = \frac{\sin AC}{\sin A\Delta} : \frac{\sin BC}{\sin B\Delta}.$$

But by hypothesis we have

$$\frac{\sin A'C'}{\sin A'D'} : \frac{\sin B'C'}{\sin B'D'} = \frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD};$$

whence,

$$\frac{\sin AC}{\sin A\Delta} : \frac{\sin BC}{\sin B\Delta} = \frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD},$$

$$\text{or, } \sin A\Delta : \sin B\Delta :: \sin AD : \sin BD,$$

a proposition which, obviously, can only hold true when Δ coincide D, or the point S′ with S, and situated in the conic section des through the other five points.

5. If from any two summits S, S of a hexagon inscribed in section, systems of lines be drawn to the remaining four summits, *a*, *b* then the angles formed by these two sets will have equal anha ratios.

For this is, obviously, merely a change of form of the enunciation fundamental principle; and, on account of its neatness of expressio *well calculated* for memorial use.

This is the celebrated property of Desargues respecting the conic section and its inscribed quadrilateral, and which gave rise to his profound inquiries concerning the principle of involution. The property itself, however, admits of proof without the aid of projective considerations, as may be seen at p. 233, vol. ii., Hutton's Course: and to say that the demonstration there given is founded on Dr. Simson's is a sufficient guarantee for its fulfilling the most rigid conditions of the ancient geometry. The method here employed is essentially that of Chasles, and is in better keeping with the spirit of the modern methods of investigation.

It should also be remarked that the *conjugate points* in this involution are those in which the *opposite sides* of the quadrilateral cut the transversal, and those in which the *same conic section* cuts it.

Likewise, that for *either* pair of sides, or for the *conic section*, the diagonals may be substituted in the enunciation. For it is clear that the former of these changes only supposes a *different order* in passing from one point to another, in the case of the four angular points: or in other words, taking a different case of the simple quadrilateral, as explained at p. 214, Hutton ii. When the diagonals are substituted for the conic section, the proposition becomes the 129th of book vii. of the Math. Coll., or prop. xvii. p. 231, Hutton ii.

XVIII.

1. If two conic sections be circumscribed about the same quadrilateral, and a transversal cut these in four points, and either pair of opposite sides in two others; these six points will form an involution:—

2. If three conic sections be circumscribed about a quadrilateral, and be cut by any transversal; the six points of section will form an involution:—

3. If n conic sections be circumscribed, *etc.*; the intersections will form an involution of $2n$ points:—

the conjugate points being in all cases those which are made by the same conic section, or the opposite pairs of sides of the inscribed quadrilateral; and the diagonals of the quadrilateral being substituted for either pair of sides or for either of the conic sections.

For generally, let a, a', b, b' be the intersections of the transversal and the opposite sides or diagonals of the quadrilateral, and let $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$, *etc.*, be the intersections of the same transversals with the successive conics circumscribed: then by the preceding

$$abaa' \text{ and } a'b'a'a,$$
$$ab\beta\beta' \text{ and } a'b'\beta'\beta,$$
$$ab\gamma\gamma' \text{ and } a'b'\gamma'\gamma,$$
$$\cdots\cdots\cdots\cdots\cdots$$

form so many involutions.

Now because $aa'bb'$ determine the centre o, of involution of six points, it will be the same for all these systems, since those four points enter into each of them. Whence

$$ao.oa' = bo.ob' = \alpha o.o\alpha'$$
$$ao.oa' = bo.ob' = \beta o.o\beta'$$
$$ao.oa' = bo.ob' = \gamma o.o\gamma'$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Wherefore,

$$\alpha o.o\alpha' = \alpha o.o\alpha' = \beta o.o\beta',$$

and the first part is proved, since this is the definition of involution.

Similarly, $\alpha o.o\alpha' = \beta o.o\beta' = \gamma o.o\gamma'$ shews that these six points are also in involution; which is the second part of the proposition.

Lastly, we get in the same way, the continuous series of equalities,

$$\alpha o.o\alpha' = \beta o.o\beta' = \gamma o.o\gamma' = \delta o.o\delta' = \ldots\ldots = \nu o.o\nu';$$

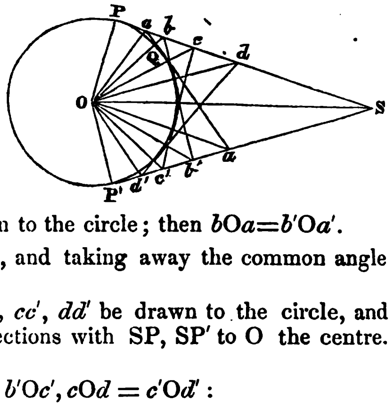
which shews that the entire system of $2n$ points forms an involution; which is the third part of the proposition.

These properties are extremely difficult to deduce by the co-ordinate method. M. Sturm has deduced the first by such means: but the complexity of his process leaves little ground for expectation that the general property expressed in the third can be deduced by this method. See Ann. des Math., tom. xvii.

XIX.

Let SP, SP′ be two tangents to a circle, and be cut by any other four tangents in a, b, c, d and a', b', c', d': then these two systems of points will have the same anharmonic ratios.

1. Let SP, SP′ touch the circle as in the enunciation, and let aa' be one of the other four tangents, and let O be the centre of the circle; and join OS, OP, OP′, Oa, Oa': then the angle aOa' is equal to SOP or SOP′.



For draw OQ to the point of contact; then $a\text{OP} = a\text{OQ}$, and $a'\text{OP}' = a'\text{OQ}$; wherefore $a'\text{OP}' + a\text{OP} = a'\text{OQ} + a\text{OQ} = a\text{O}a'$. But $a'\text{OP}' + a\text{OP} = \frac{1}{2}(\text{POQ}+\text{P'OQ}) = \frac{1}{2}\text{POP}' = \text{SOP} = \text{SOP}'$: and hence $a\text{O}a' = \text{SOP} = \text{SOP}'$.

2. Let another tangent bb' be drawn to the circle; then $b\text{O}a = b'\text{O}a'$.

For by the preceding, $a\text{O}a' = b\text{O}b'$, and taking away the common angle $b\text{O}a'$ we have $b\text{O}a = b'\text{O}a'$.

3. Let four such tangents aa', bb', cc', dd' be drawn to the circle, and draw lines from each of their intersections with SP, SP′ to O the centre. Then by the preceding we have

$$a\text{O}b = a'\text{O}b',\ b\text{O}c = b'\text{O}c',\ c\text{O}d = c'\text{O}d':$$

and hence the anharmonic ratios of these angles is the same. Wherefore also the anharmonic ratios of their sections with SP and SP′ are the same: or the system $abcd$ has the same anharmonic ratios as the system $a'b'c'd'$. The proposition is, therefore, proved.

XX.

1. If SP, SP′ be two tangents to a conic section, and any four other tangents aa', bb', cc', dd' be drawn to intersect these: then the anharmonic ratios of $abcd$ will be equal to those of $a'b'c'd'$.

For the conic section may be considered as the projection of a circle, and its tangents as the projections of those of the circle. If then we denote by $\alpha\beta\gamma\delta$ and $\alpha'\beta'\gamma'\delta'$ the points on the two tangents $\Sigma\pi$, $\Sigma\pi'$ of the circle: then

the anharmonic ratios of $abcd$ are the same with those of $\alpha\beta\gamma\delta$, and those of $a'b'c'd'$ the same with those of $\alpha'\beta'\gamma'\delta'$; and as the anharmonic ratios of $\alpha\beta\gamma\delta$ and $\alpha'\beta'\gamma'\delta'$ are the same (by XIX. 3.) we have also the equality of anharmonic ratios stated in the proposition. Or in symbols :—

$$\frac{\alpha\gamma}{\alpha\delta} : \frac{\beta\gamma}{\beta\delta} = \frac{ac}{ad} : \frac{bc}{bd},$$

$$\frac{\alpha'\gamma'}{\alpha'\delta'} : \frac{\beta'\gamma'}{\beta'\delta'} = \frac{a'c'}{a'd'} : \frac{b'c'}{b'd'},$$

$$\text{and } \frac{\alpha\gamma}{\alpha\delta} : \frac{\beta\gamma}{\beta\delta} = \frac{\alpha'\gamma'}{\alpha'\delta'} : \frac{\beta'\gamma'}{\beta'\delta'}.$$

Hence, also, we have

$$\frac{ac}{ad} : \frac{bc}{bd} :: \frac{a'c'}{a'd'} : \frac{b'c'}{b'd'}.$$

Conversely :—When two straight lines, situated in the same plane, are divided each in four points, such that the anharmonic ratios of one are equal to the corresponding ratios of the other, the four lines joining these points, and the two lines in which the points are situated, form six tangents to the same conic section.

For a conic section may be described to touch any five lines: let this be drawn to touch SP, SP′, aa', bb', and cc'. Then if it do not also touch dd', let a tangent be drawn from d to the conic section, intersecting SP′ in δ. Then by the proposition $a'b'c'\delta$ will have the same anharmonic ratio with $abcd$; and by hypothesis $abcd$ has the same anharmonic ratio with $a'b'c'd'$. Hence $a'b'c'd'$ and $a'b'c'\delta$ have the same anharmonic ratio; that is

$$\frac{a'c'}{a'd'} : \frac{b'c'}{b'd'} = \frac{a'c'}{a'\delta} : \frac{b'c'}{b'\delta},$$

$$\text{or} \qquad b'd' : a'd' :: b'\delta : a'\delta,$$

which is a manifest impossibility except δ and d' coincide. The sixth line dd' is therefore a tangent to the conic section. Whence the six lines are tangents to the same conic section, as stated in the enunciation.

These properties together with that in (XVII. 3.), constitute the basis of the application of anharmonic ratios and involution to the conic sections: and to understand the force of Chasles's reasonings it is essential that these two properties should be carefully considered and fully comprehended. For some classes of inquiry, however, they are found to be more conveniently put into other forms.

XXI.

1. Let us conceive that the three lines of the first system A, B, C, and the three corresponding ones of the second system A′, B′, C′, in (XVII) be fixed, whilst the fourth of each system is so changed in its position, as to still retain the equality of the anharmonic ratios of the two systems: then their mutual intersection will always be situated on the conic section determined by its passing through the other five points.

For it will trace out a conic section which passes through the other five points by (XVII. 4); and as only one conic section can be made to pass through five points, it must coincide with that traced by the said intersection.

The practical objects of this system being more directly those of geometrical construction than of algebraic calculation, it will be unnecessary to dwell, with any considerable detail, upon the algebraical forms of expression which result; but rather upon such as offer the best modes of construction and of deducing the properties. We may however remark that even if calculation were our great object, it could be very simply effected. For, if D′ were the line required to form the sixth intersection, we should readily find it from

$$\frac{\sin D'A'}{\sin D'B'} = \frac{\sin DA}{\sin DB} \cdot \frac{\sin CB}{\sin CA} \cdot \frac{\sin C'A'}{\sin C'B'},$$

which is only a slight transformation of the anharmonic equation.

2. Let us conceive that the four sides of a given quadrilateral (or two opposite sides and the two diagonals), cut, in four points, a variable transversal which is subjected to pass through a fifth given point, and that in this transversal a sixth point be taken, so that the six points shall form an involution, then this sixth point will always be in a conic section which passes through the four summits of the quadrilateral, and the fifth given point.

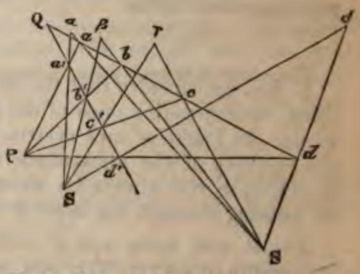
For, a conic section can always be drawn through the summits of the quadrilateral and the fifth point; and this will by its intersection with the transversal complete the involution: and as through the five points only one conic section can be described, it is impossible for the sixth involution point of the construction to be any other than the second intersection of the curve with the transversal—or, in other words, for it not to be in the curve of the conic section.

It is upon this that the construction of the conic sections by point, was founded by Desargues. The algebraic expression of the point, were it required, may in each case be obtained, either from any one of the three equations of involution of Desargues, or from any one of the four of Pappus: as for instance, if c' be the sixth point, and we take the first property of Pappus we have

$$\frac{bc'}{ac'} = \frac{cb'.ba'}{c'b.b'a}.$$

Not only many modes of construction may be deduced from these two statements of the properties, but also the most general properties of the curves themselves.

XXII.



1. Let Qd, Qd' be any two lines meeting in Q, and from any point P let there be drawn lines to cut these in a, b, c, d, and a', b', c', d' respectively; also from any other two points S and S′ draw lines Sa, Sb, Sc, Sd, $S'a'$, $S'b'$, $S'c'$, $S'd'$: then the anharmonic ratios of the angular systems at S and S′ will be equal.

For denoting these lines as before by A, B, C, D, and A′, B′, C′, D′; we have from (II, 2.)

$$\frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD} = \frac{ac}{ad} : \frac{bc}{bd},$$

$$\frac{\sin A'C'}{\sin A'D'} : \frac{\sin B'C'}{\sin B'D'} = \frac{a'c'}{a'd'} : \frac{b'c'}{b'd'},$$

and $$\frac{ac}{ad} : \frac{bc}{bd} = \frac{a'c'}{a'd'} : \frac{b'c'}{b'd'} :$$

whence,

$$\frac{\sin AC}{\sin AD} : \frac{\sin BC}{\sin BD} = \frac{\sin A'C'}{\sin A'D'} : \frac{\sin B'C'}{\sin B'D'}.$$

2. If the corresponding lines Sa and S$'a'$, Sb and S$'b'$, Sc and S$'c'$, and Sd and S$'d'$ intersect in $\alpha, \beta, \gamma, \delta$; the points $\alpha, \beta, \gamma, \delta$, Q, S and S$'$ will be in the same conic section.

For the construction renders $abcd$ and $a'b'c'd'$ in equal anharmonic ratios, and the systems of radiation from S and S$'$ in equal anharmonic ratios, as just proved. If, then, we consider S and S$'$ as opposite summits of a quadrilateral, and γ, δ as the other two summits; and if we consider β to be the fifth point through which with S, S$'$, γ, δ the conic section is made to pass: then by (XVII. 4) the fourth point α in which Sa, Sa' intersect, will also be in the same conic section.

It only remains to shew that Q is also in the same conic section.

But this is evident from the consideration that as the line Paa' is arbitrary in its position, the equality of the anharmonic ratios is not disturbed by any change in its position. It may therefore be drawn through Q, in which case a coincides with a', and hence with the intersection Q of Sa and S$'a'$.

3. When the three sides of a triangle of variable form revolve about three fixed points, and two of its summits move upon two fixed lines; then the third summit traces a conic section which passes through those two of the fixed points about which the two sides adjacent to that summit revolve, and likewise through the intersection of the fixed lines.

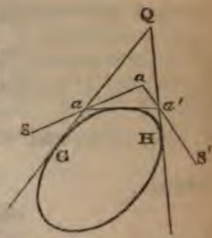
For the conic section in the last case is generated by α the summit of the triangle $a\alpha a'$, the three sides of which pass through P, S, S$'$, and the two summits a', a of which are situated on the fixed lines Qa, Qa'; and this conic section passes also through S, S$'$ the two fixed poles about which the sides adjacent to the summit α turn, and through Q the point in which the fixed lines Qa, Qa' meet.

This last theorem was claimed both by Maclaurin and Braikenridge; and is a very important one, both in reference to the theory and the construction of the conic sections. Priority of publication entitles Braikenridge to the honor of having his name attached to it: especially as there is every reason to believe that he was the first to discover the *principle* upon which such modes of genesis are founded in the ordinary way of treating the subject. At the same time there is not the slightest cause to question Maclaurin's claim to being an independent discoverer of the same theorem and the same class of methods.

Another demonstration of it may be seen in Hutton, ii. p. 188.

XXIII.

Let QG, QH be any two given tangents to a given conic section, and S, S′ any two given points; and if a variable triangle *aaa′* have one of its sides a tangent to the conic section, and its two others revolving about S and S′, whilst its two summits *a*, *a′* move upon QG, QH; then the third summit *a* will trace out another conic section passing through S, S′ and Q.



For, it is clear from the reasoning of the preceding general proposition, that the only condition respecting the divisions of the two lines QG, QH, in which *a*, *a* are situated, that is implied in the premises requisite to cause *a* to describe a conic section, is—that they shall be divided in the same anharmonic ratio. But this, by (XIX) is the case with *any four tangents* to the given conic section: and hence the conclusion follows at once.

This theorem is more general than that of Braikenridge, and, as Chasles remarks, "gives rise to a great number of propositions, the greater part of which are new." It will be a useful exercise to the young geometer to deduce some of them.

XXIV.

We shall pass over some of Chasles's further illustrations, the present being deemed sufficient for the purpose of shewing the method of the application of the principles to *lines of the second order:* and on a future occasion give a series of applications of the same principles to the corresponding surfaces—and especially to the deduction of the most general properties of the two rule-surfaces, the hyperbolic paraboloïd and the hyperboloïd of one sheet, which are always found to be the most intractable by the co-ordinate method. They are, however, more simple by the present method than even the surfaces which are curved in all directions.

It ought here to be remarked that the proof offered by Chasles of Newton's Organic process for describing lines of the second order, appears to be unsatisfactory; and that his generalisation of Newton's theorem is certainly not true—as evinced by the application of the co-ordinate method giving to the curve described, according to Chasles's process, an equation of the fourth degree. It is, indeed, barely possible that some method of reduction may be found which will reduce this equation to two factors of the second degree; and in addition to this, that one of these factors may be shewn to be foreign to the inquiry. In such case the conclusion above stated will be erroneous: but I have taken every precaution for the detection of such a circumstance, and feel some degree of confidence that my conclusion will be found correct. At any rate, even if Chasles's generalisation be correct, his *reasoning* does not warrant the conclusion: there being an oversight in the early part which vitiates all the subsequent steps.

To this kind of mistake, all *general reasoning* is peculiarly liable; and it becomes of the utmost importance in the class of investigations which constitutes the *modern Geometry*, to be upon our guard against it. Since the time of Dr. Matthew Stewart, there have probably existed only two geometers possessed of the remarkable power of "thinking out" a process, to the extent that he did: — Monge and Chasles. Of the three, Monge alone is the one *who has not*, as far as is known, committed some oversight or other.

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To find the locus of the centres of circles tangential to two given circles.

Before proceeding to an investigation, it is perhaps necessary to observe, that the variable circles can be tangential, to the given ones, in different ways; by their being convex or concave to both, or convex to the one and concave to the other, and vice versa. I shall however take as the first case, that where the touching circles are convex to the given ones, and these exterior to one another.

Let A,B be the centres of the given circles, r, r' their respective radii, a the distance between A and B; and suppose $r > r'$. Take A, for the origin of rectangular axes, AB for axis of x; also denote the co-ordinates of O, the centre of one of the variable circles by xy, and its radius by R. Then

$$\mathrm{R}+r=\sqrt{x^2+y^2},\ \mathrm{R}+r'=\sqrt{y^2+(a-x)^2};$$

whence $$\sqrt{x^2+y^2}-\sqrt{y^2+(a-x)^2}=r-r'.$$

Squaring, reducing, etc., this becomes

$$(r-r')^2y^2-\{a^2-(r-r')^2\}x^2+a\{a^2-(r-r')^2\}x-\tfrac{1}{4}\{a^2-(r-r')^2\}^2=0\ ..(1)$$

Comparing (1) with the general equation of the second degree, we find $\mathrm{B}^2-4\mathrm{AC}>0$; the locus is therefore an hyperbola, so long as $a>(r-r')$, and its foci are A,B. When circle (A) touches (B) externally: then $a=r+r'$, and (1) reduces to

$$\tfrac{1}{4}(r-r')^2y^2-rr'x^2+rr'(r+r')x-r^2r'^2=0\ ....(2)$$

the equation to an hyperbola; and when they intersect, the locus is again an hyperbola, for $a>(r-r')$. When $a=r-r'$, that is, when circle (B) touches (A) internally, (1) becomes $y^2=0$; which designates the axis of x. The variable circles will in this case, envelope one another; and hence their centres will be in a line coinciding with AB.

Let now the variable circles be convex to (A) and concave to (B); then in this case, $\mathrm{OA}=\mathrm{R}+r$, and $\mathrm{OB}=\mathrm{R}-r'$; or when concave to (A) and convex to (B), $\mathrm{OA}=\mathrm{R}-r$, and $\mathrm{OB}=\mathrm{R}+r'$: whence $\mathrm{OA}-\mathrm{OB}=r+r'$, or $\mathrm{OB}-\mathrm{OA}=r+r'$. Operating as before, we have the equation

$$(r+r')^2y^2-\{a^2-(r+r')^2\}x^2+a\{a^2-(r+r')^2\}x+\tfrac{1}{4}\{a^2-(r+r')^2\}^2=0\ ..(3)$$

whose locus is an hyperbola, so long as $a>(r+r')$; that is, when the given circles are exterior to one another: but when they touch externally, (3) reduces to $y^2=0$, the equation to the axis of x.

If we suppose circle (B) to touch (A) internally; then $a>(r+r')$, and the locus of the variable centres is an ellipse, whose foci are A,B. It may, however, be shown otherwise: for since in this case, $\mathrm{AO}=r-\mathrm{R}$, and $\mathrm{BO}=r'+\mathrm{R}$; whence $\mathrm{AO}+\mathrm{OB}=r+r'$, the characteristic property of the ellipse. When the given circles are concentric, the locus is a circle; for we find

$$y^2+x^2=\tfrac{1}{4}(r+r')^2.$$

In the case, where the variable circles are concave to the given ones, it may be easily shown, that the locus of their centres is an hyperbola. There are also other cases, such as when the given circles are equal, and exterior to one another; or when they touch externally, or intersect one another. Also when r or r' is $=0$, or when at the same time $r=0$ and $r'=0$: cases too simple in themselves to dwell upon here, but which may be readily deduced from equations (1) and (3).

APPLICATION OF ALGEBRA TO THE MODERN GEOMETRY.

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"On doit, dans toutes parties des sciences, accoutumer l'esprit à toujours établir ses spéculations sur vérités les plus générales que présente chaque théorie. C'est le plus sûr, si non l'unique moyen de simplifier l'étude d'une science et d'en assurer les progrés."—CHASLES.

SECTION I.

ON THE CONTACT AND INTERSECTION OF PLANE CURVES.

Containing extensions of the theories of the double contact of conic sections, and the involution of six points.

I.

If the equations

$$\alpha y+\beta x=1 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\alpha' y+\beta' x=1 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

denote any two chords (real or ideal) of the conic section

$$\mathrm{A}y^2+2\mathrm{B}xy+\mathrm{C}x^2+2\mathrm{D}y+2\mathrm{E}x+1=0 \ldots\ldots\ldots\ldots (3)$$

and if m be an arbitrary constant; the equation

$$\mathrm{A}y^2+2\mathrm{B}xy+\mathrm{C}x^2+2\mathrm{D}y+2\mathrm{E}x+1=m(\alpha y+\beta x-1)(\alpha' y+\beta' x-1) \ldots (4)$$

will represent a conic section passing through the points (real or imaginary) in which the straight lines (1) and (2) intersect the curve (3). For, by taking the difference of equations (3) and (4), we get

$$m(\alpha y+\beta x-1)(\alpha' y+\beta' x-1)=0;$$

from which it is evident that the straight lines (1) and (2) are conjugate common secants of the curves (3) and (4).

II.

When the straight lines (1) and (2) coincide, the points in which the curves (3) and (4) intersect evidently coalesce in two points of contact. In this case equation (4) becomes

$$\mathrm{A}y^2+2\mathrm{B}xy+\mathrm{C}x^2+2\mathrm{D}y+2\mathrm{E}x+1=m(\alpha y+\beta x-1)^2,$$

or, by changing the forms of the last three constants,

$$\mathrm{A}y^2+2\mathrm{B}xy+\mathrm{C}x^2+2\mathrm{D}y+2\mathrm{E}x+1=(ay+bx-c)^2 \ldots\ldots (5)$$

hence this equation represents a conic section having a double contact (real or imaginary) with the curve (3). The straight line (1), or

$$ay+bx-c=0 \ldots\ldots (1'),$$

evidently passes through the two points of contact, and is therefore called the *chord of contact*.

When the chord (1) becomes a tangent to the curve (3), the two points of contact are united in one: consequently, in this case, the curves (3) and (5) have a contact of the third order; and the straight line (1) is a common tangent at the point of contact.

III.

It is evident from what has been advanced that the equation

$$\mathrm{A}y^2+2\mathrm{B}xy+\mathrm{C}y^2+2\mathrm{D}y+2\mathrm{E}x+1=(a'y+b'x-c')^2 \ldots (6)$$

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represents a conic section, having a double contact with the curve (3), on the straight line

$$a'y+b'x-c'=0\ \ldots\ldots (2').$$

Now by subtracting (5) from (6) we get

$$(a'y+b'x-c')^2-(ay+bx-c)^2=0:$$

which may be resolved into the two equations

$$(a'+a)y+(b'+b)x=c'+c\ \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$(a'-a)y+(b'-b)x=c'-c\ \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

hence the intersections of the curves (5) and (6) range upon the two straight lines (7) and (8): and since the straight lines (1′), (2′), (7), (8) pass through the same points, and form a harmonic pencil, we have the following theorem.

If each of two conic sections have a double contact with a third conic section, two of their conjugate common secants pass through the point of intersection of the chords of contact, and form with these chords a harmonic pencil.

When the straight lines (1′) and (2′) become tangents to the curve (3), the double contacts unite in two points of contact of the third order (II). Hence, *if two conic sections have each a contact of the third order with another conic section, two of their conjugate common secants pass through the point of intersection of the common tangents applied at the points of contact, and form with these lines a harmonic pencil.*

It is evident also that the theorem is true when one of the curves (5) or (6) has a double contact with (3), and the other a single contact of the third order.

IV.

If a conic section have a double contact with a given conic section, and pass through two given points, the chord of contact will pass through a given point.

Let the given conic section be denoted by

$$u=Ax^2+2Bxy+Cx^2+2Dy+2Ex+1=0,$$

and the chord of contact by

$$ay+bx-c=0;$$

then the equation of the other conic section will be

$$u=(ay+bx-c)^2:$$

and if $x'y'$, $x''y''$ be the given points, we shall have

$$ay'+bx'-c=u'^{\frac{1}{2}},$$

$$ay''+bx''-c=u''^{\frac{1}{2}}.$$

From these we obtain, by eliminating the last terms,

$$a(y'u''^{\frac{1}{2}}-y''u'^{\frac{1}{2}})+b(x'u''^{\frac{1}{2}}-x''u'^{\frac{1}{2}})-c(u''^{\frac{1}{2}}-u'^{\frac{1}{2}})=0;$$

from which it appears, that the chord of contact,

$$ay+bx-c=0,$$

passes through the points whose co-ordinates are

$$\frac{x'u''^{\frac{1}{2}}-x''u'^{\frac{1}{2}}}{u''^{\frac{1}{2}}-u'^{\frac{1}{2}}} \text{ and } \frac{y'u''^{\frac{1}{2}}-y''u'^{\frac{1}{2}}}{u''^{\frac{1}{2}}-u'^{\frac{1}{2}}}.$$

Hence it is evident that *if a system of conic sections have each a double contact with a given conic section, and if each of them pass through two given points, the chords of contact will pass through the same point.*

In like manner it may be shown that *if a conic section have a contact of the third order with a given conic section, and pass through two given points, the common tangent applied at the point of contact will pass through a given point.*

It is evident also that a conic section may be described so as to pass through three given points, and have a double contact, or a single contact of the third order, with a given conic section.

It is easy to show that the point of intersection of the chords of contact lies on the straight line joining the given points $x'y'$, $x''y''$.

V.

The conic section represented by the equation

$$Ay^2+2Bxy+Cx^2+2Dy+2Ex+1=ay+bx-c \quad\ldots\ldots\ldots\ldots(9)$$

will pass through the two points (real or imaginary) in which the chord (1′) (real or ideal) cuts the curve (3). For by subtracting (3) from (9) we get

$$ay+bx-c=0;$$

and therefore the intersections of the curves (3) and (9) lie on the straight line (1′). Hence the curves (3) and (9) can intersect each other only in two points which are both real or both imaginary.

In like manner it may be shown that the conic section

$$Ay^2+2Bxy+Cx^2+2Dy+2Ex+1=a'y+b'x-c' \quad\ldots\ldots\ldots\ldots(10)$$

passes through the points in which the chord (2′) cuts the conic section (3). Now by subtracting (9) from (10) we get

$$(a'-a)y+(b'-b)x=c'-c \quad\ldots\ldots\ldots\ldots\ldots\ldots(11);$$

and consequently the curves (9) and (10) intersect in two points (real or imaginary) which lie on the chord (11) (real or ideal): and since the chord (11) passes through the intersection of (1′) and (2′) we have the following theorem.

If each of two conic sections meet a third conic section in two points only, they intersect each other in two points only; and the three common secants pass through the same point.

When the chords $ay+bx-c=0$, and $a'y+b'x-c'=0$, become tangents to the curve (1), the common secant (11) of the curves (9) and (10) will evidently pass through the intersection of the tangents.

Since equation (9) contains only three arbitrary constants, it appears that if a conic section be restricted so as to cut a given conic section in two points only, it can only be subjected to three other conditions; such as passing through three given points, *etc.* This result is very remarkable.

VI.

If a conic section pass through two given points, and intersect a given conic section in two points only, the common secant will pass through a fixed point.

Let $u=0$ be the equation of the given conic section, $x'y'$ and $x''y''$ the given points, and

$$ay+bx-c=0,$$

the equation of the common secant: then the equation of the other conic section will be (v)

$$u = ay + bx - c;$$

and, since the curve passes through the two given points, we shall have

$$ay' + bx' - c = u',\ ay'' + bx'' - c = u''.$$

By multiplying the first of these equations by u'', the second by u', and subtracting, we get

$$a(y'u'' - y''u') + b(x'u'' - x''u') - c(u'' - u') = 0;$$

hence the chord $ay + bx - c = 0$, passes through the point

$$\frac{x'u'' - x''u'}{u'' - u'},\ \frac{y'u'' - y''u'}{u'' - u'};$$

which evidently lies on the straight line joining the two given points.

Hence *if a system of conic sections pass through two given points, and if each of them intersect a given conic section in two points only, the common secants will pass through a given point in the straight line joining the two given points; and if one of them touch the given conic section the common tangent will pass through the same point.*

VII.

A point (A), a conic section (C), and two straight lines MN, M'N' being given, it is required to find the locus of a point (P), such that if PA be drawn meeting the conic section in Q, Q'; and if PM, PM' be drawn perpendicular to the given lines; PQ×PQ' shall be to AQ×AQ' as PM×PM' is to a given space.

The point A being taken as the origin of rectangular co-ordinates, let the equations of the two straight lines and of the conic section be

$$\alpha y + \beta x = 1 \ldots\ldots\ldots (a) \qquad \alpha' y + \beta' x = 1 \ldots\ldots\ldots (b)$$

$$Ay^2 + 2Bxy + Cx^2 + 2Dy + 2Ex + 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (c)$$

Put AP $= r$, xAP $= \theta$, AQ $= \rho'$, AQ' $= \rho''$: then if $x'y'$ be the co-ordinates of P, we shall have

$$PM = \frac{\alpha y' + \beta x' - 1}{\sqrt{(\alpha^2 + \beta^2)}},\quad PM' = \frac{\alpha' y' + \beta' x' - 1}{\sqrt{(\alpha'^2 + \beta'^2)}};$$

hence by the condition of the question

$$\frac{(r - \rho')(r - \rho'')}{\rho'\rho''} = m\,(\alpha y' + \beta x' - 1)(\alpha' y' + \beta' x' - 1),$$

m being a given quantity; or

$$\frac{r^2}{\rho'\rho''} - r\left(\frac{1}{\rho'} + \frac{1}{\rho''}\right) + 1 = m(\alpha y' + \beta x' - 1)(\alpha' y' + \beta' x' - 1) \ldots (d)$$

Now the polar equation of the ellipse being

$$(A \sin^2\theta + 2B \sin\theta \cos\theta + C \cos^2\theta)\rho^2 + 2(D \sin\theta + E \cos\theta)\rho + 1 = 0;$$

since ρ', ρ'' are the roots of this equation, we obtain by quadratics

$$\frac{1}{\rho'\rho''} = A \sin^2\theta + 2B \sin\theta \cos\theta + C \cos^2\theta,\quad \frac{1}{\rho'} + \frac{1}{\rho''} = -2(D \sin\theta + E \cos\theta);$$

hence, by substitution, equation (d) becomes

$$r^2(A \sin^2\theta + 2B \sin\theta \cos\theta + C \cos^2\theta) + 2(D \sin\theta + E \cos\theta)r + 1$$
$$= m(\alpha y' + \beta x' - 1)(\alpha' y' + \beta' x' - 1);$$

but $r\sin\theta=y'$ and $r\cos\theta=x'$; therefore

$$Ay'^2+2Bx'y'+Cx'^2+2D'y'+2Ex'+1=m(\alpha y'+\beta x'-1)(\alpha' y'+\beta' x'-1).(e)$$

consequently the required locus is a conic section, passing through the points (real or imaginary) in which the straight lines (a) and (b) meet the curve (c).

This proposition, which is easily extended to algebraic curves of every degree, exhibits a very simple geometrical relation between the lines (1), (2), (3), (4) of No. I.

In like manner it may be shown that the curve (9) is the locus of a point (P), such that if a straight line be drawn from it to the origin (A), meeting the curve (3) in Q and Q′, and a perpendicular PM to the straight line (1); then PQ×PQ′ : AQ×AQ′ : : PM : a given straight line.

VIII.

Let the equations

$$\alpha y+\beta x=1\ldots\ldots(12)$$
$$\alpha' y+\beta' x=1\ldots\ldots(13)$$
$$\alpha'' y+\beta'' x=1\ldots\ldots(14)$$

denote three chords (real or ideal) of the curve

$$u=Ay^3+Bxy^2+Cx^2y+Dx^3+Ey^2+Fxy+Gx^2+Hy+Kx+1=0\ \ldots(15)$$

then the equation

$$u=m(\alpha y+\beta x-1)(\alpha' y+\beta' x-1)(\alpha'' y+\beta'' x-1)\ldots\ldots(16)$$

will represent a curve of the third degree passing through the nine points (real or imaginary) in which the straight lines (12), (13), (14) meet the curve (15). For by subtracting (15) from (16) we get

$$m(\alpha y+\beta x-1)\ (\alpha' y+\beta' x-1)\ (\alpha'' y+\beta'' x-1)=0:$$

hence the points of intersection of the curves (15) and (16) lie on the straight lines (12), (13), (14).

IX.

When the straight lines (12), (13), (14) coincide, the nine points of intersection unite in (3) points of contact, which are therefore of the second order. Hence the equation

$$u=m(\alpha y+\beta x-1)^3,\text{ or}$$
$$u=(ay+bx-c)^3\ldots\ldots(17)$$

represents a curve of the third degree, having a triple contact of the second order with the curve (15), at three points which lie in the straight line

$$ay+bx-c=0\ldots\ldots(12')$$

As in No. II this straight line may be called the *chord of contact.*

If

$$a'y+b'x-c'=0\ldots\ldots(13')$$

be any other chord of the curve (15), the equation of a curve of the third degree, having a triple contact of the second order with (15) on the straight line (13′) is

$$u=(a'y+b'x-c')^3\ldots\ldots(18)$$

hence, by subtracting (17) from (18), we get

$$(ay+bx-c)^3-(a'y+b'x-c')^3=0,$$

which may be resolved into the two equations

$$(a-a')y+(b-b')x-(c-c')=0\ \ldots\ldots(19)$$
$$(ay+bx-c)^2+(ay+bx-c)\,(a'y+b'x-c')+(a'y+b'x-c')^2=0\ldots(20)$$

consequently *if two curves of the third degree have each a triple contact of the second order with a given curve of the third degree, at three points which lie in the same straight line, the points in which they intersect lie upon the same straight line* (19), *which evidently passes through the intersection of the chords of contact* (12′) *and* (13′).

X.

It is evident from what has been said in the last two numbers that the equation

$$u=(ay+bx-c)^2 \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

denotes a curve of the third degree, having a triple contact of the first order with (15) on the straight line (12′). Similarly, the equation

$$u=(a'y+b'x-c')^2 \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

represents a curve of the third degree, having a triple contact of the first order with (15) on the chord (13′). But by subtracting (21) from (22) we get

$$(a'y+b'x-c')^2-(ay+bx-c)^2=0;$$

which is resolvable into the two equations

$$(a'+a)y+(b'+b)x=c'+c \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

$$(a'-a)y+(b'-b)x=c'-c \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

hence *if any two curves of the third degree have each a triple contact of the first order with a given curve of the third degree, at three points which lie in the same straight line, the points in which they intersect each other range upon two straight lines, which pass through the intersection of the chords of contact, and form with them a harmonic pencil.*

XI.

Again, the equations

$$u=ay+bx-c \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

$$u=a'y+b'x-c' \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

evidently denote two curves of the third degree, each intersecting the curve (15) in three points only, which lie on the straight lines (12′) and (13′) respectively. But, by subtracting (25) from (26), we obtain

$$(a'-a)y+(b'-b)x=c'-c \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

which denotes a straight line passing through the intersection of (12′) and (13′): hence, *if each of two curves of the third degree intersect a given curve of the third degree in three points only, which lie in a straight line, they intersect each other in three points only, which lie in a straight line; and the three common secants pass through the same point.*

XII.

If a curve of the third degree pass through two given points, and intersect a given curve of the third degree in three points only which lie in a straight line, the common secant will pass through a given point in the straight line joining the two given points.

Let $u=0$ denote the given curve; $x'y'$ and $x''y''$ the given points,

$$ay+bx-c=0$$

the common secant: then, by the question, we obtain (25)

$$ay'+bx'-c=u',$$

$$ay''+bx''-c=u'';$$

$$\therefore\; a(y'u''-y''u')+b(x'u''-x''u')-c(u''-u')=0:$$

consequently the secant $ay+bx-c=0$ passes through the point

$$\frac{x'u''-u'x''}{u''-u'},\quad \frac{y'u''-u'y''}{u''-u'};$$

which evidently lies on the straight line $y-y'=\dfrac{y'-y''}{x'-x''}(x-x')$.

Hence, *in a system of curves of the third degree, if each curve of the system intersect a given curve of the third degree in three points only which lie in a straight line, and pass through two given points, the common secants will pass through the same point.*

In like manner it follows from equations (17) and (21) that *if a system of curves of the third degree have each a triple contact of the first or second order with a given curve of the third degree at three points which lie in a straight line; and if each of them pass through two given points, the chords of contact will pass through a given point.*

XIII.

The above theorems may now be easily extended to curves of every degree. Thus if $u=0$ be a curve of the nth degree, and if

$$\left.\begin{array}{r} \alpha_1 y+\beta_1 x=1 \\ \alpha_2 y+\beta_2 x=1 \\ \dots\dots\dots\dots \\ \alpha_m y+\beta_m x=1 \end{array}\right\} \dots\dots\dots\dots\dots\dots (a)$$

be m chords (real or ideal) of the curve $u=0$, m being not greater than n; then the equation

$$u=q(\alpha_1 y+\beta_1 x-1)(\alpha_2 y+\beta_2 x-1)\dots(\alpha_m y+\beta_m x-1)\dots\dots (b)$$

represents a curve of the n^{th} degree passing through the mn points (real or imaginary) in which the chords (a) meet the curve $u=0$. When the straight lines (a) coincide, the mn points of intersection unite in n points of contact: hence the equations

$$\left.\begin{array}{l} u=(a\,y+b\,x-c\,)^m \\ u=(a'y+b'x-c')^m \end{array}\right\} \dots\dots\dots\dots\dots (c)$$

denote two curves of the n^{th} degree having a contact of the $(m-1)^{\text{th}}$ order with the curve $u=0$, at n points which lie on the chords of contact

$$\left.\begin{array}{l} a\,y+b\,x=c \\ a'y+b'x=c' \end{array}\right\} \dots\dots\dots\dots\dots\dots (d)$$

Now, from equations (c), we obtain by subtraction

$$(a'y+b'x-c')^m-(ay+bx-c)^m=0;$$

which, when m is odd, is equivalent to the equation

$$(a'-a)y+(b'-b)x=c'-c$$

and the simultaneous equations (d); to which the equation

$$(a'+a)y+(b'+b)x=c'+c,$$

must be added when m is even. Hence *if two curves of the n^{th} degree touch a given curve of the n^{th} degree at n points in the same straight line, their intersections range upon two straight lines or upon one straight line passing through the intersection of the chords of contact according as the contact is of an odd or even order.*

Again, if a system of curves of the n^{th} degree pass through two given points, and if each of them touch a given curve of the n^{th} degree at n points in the same straight line, we should find, as in No. III, that the chords of contact must pass through a given point in the straight line joining the two given points.

Also, it may be shown, as in No. VII, that the curve (b) is the locus of a point such that if a straight line be drawn from it to the origin A, meeting the curve $u=0$ in $Q_1, Q_2, Q_3 \ldots Q_n$; and if perpendiculars $PM_1, PM_2 \ldots PM_m$ be drawn to the straight lines (a): then

$$PQ_1 \times PQ_2 \ldots \times PQ_n : AQ_1 \times AQ_2 \ldots \times AQ_n :: PM_1 \times PM_2 \ldots \times PM_m : a^m,$$

a being a given line.

XIV.

If $u=0$ be a curve of the n^{th} degree, the equation

$$u=(ay+bx-c)^p\,(a'y+b'x-c')^q \ldots\ldots\ldots\ldots(a)$$

(where $p+q<n$) evidently denotes a curve of the n^{th} degree, having a contact of the $(p-1)^{\text{th}}$ order with the curve $u=0$ at n points on the chord

$$ay+bx-c=0 \ldots\ldots\ldots\ldots(b)$$

and a contact of the $(q-1)^{\text{th}}$ order on the chord

$$a'y+b'x-c'=0 \ldots\ldots\ldots\ldots(c)$$

Again the equation

$$u=(a'y+b'x-c')^p\,(ay+bx-c)^q \ldots\ldots\ldots\ldots(d)$$

represents a curve of the n^{th} degree, having with the curve $u=0$ a contact of the $(p-1)^{\text{th}}$ order on the chord (c), and one of the $(q-1)^{\text{th}}$ order on the chord (b). But by subtracting (a) from (d) we get

$$(a'y+b'x-c')^p\,(ay+bx-c)^q-(ay+bx-c)^p\,(a'y+b'x-c')^q=0, \text{ or}$$

$$(ay+bx-c)^q\,(a'y+b'x-c')^q\,\{(a'y+b'x-c')^{p-q}-(ay+bx-c)^{p-q}\}=0$$

which, when $p-q$ is odd may be resolved into

$$(ay+bx-c)^q=0,\ (a'y+b'x-c')^q=0,$$

$$(a'-a)y+(b'-b)x-(c'-c)=0 \ldots\ldots\ldots\ldots(e).$$

Hence the curves (a) and (d) have a contact of the $(q-1)^{\text{th}}$ order at $2n$ points, which range on the straight lines (b) and (c), and cut each other in n points on the straight line (e).

When $p-q$ is even, the curves (a) and (d) also cut each other in n points on the straight line

$$(a'+a)y+(b'+b)x-(c'+c)=0 \ldots\ldots\ldots\ldots(e')$$

This proposition may be easily generalized.

(To be continued.)

ON THE DECOMPOSITION OF RATIONAL FRACTIONS, AND THE SUMMATION OF INFINITE SERIES.

[*Mr. Robert Rawson, Manchester.*]

The application of the decomposition of rational fractions to the summation of infinite series, integrating and effecting the general differentiation of ·e which contain several factors in their denominators, renders it of great ·ance in the various branches of analysis.

In obtaining equation (2), which is in a form well adapted for practice, it will be seen that I have employed the same mode of reasoning which is generally adopted in the investigation of Maclaurin's or more properly, Stirling's Theorem; and also the well known artifice of taking a particular state of the function in order to determine the values of the arbitrary constants.

The table containing the values of the $f(a)$ and its derived functions will, I believe, facilitate the computation of the constants used in decomposing fractions of the form $\frac{1}{(x-a)^n\,(x-b)^p\,(x-c)^q\ldots}$, and thereby render the integration of a large class of algebraical functions a matter of but ordinary difficulty.

The notation which I have adopted in Article 3 is different from that which is generally used in such inquiries; but if the distinctness and comprehensiveness of the idea which it involves will justify the use of the symbol by which it is represented, I am persuaded that it will be found, on examination, to possess a just claim to our approval.

Thus $\int_a^b f(x)dx$ represents the summation of the series whose general term is $f(x)$, where x takes every possible value between the limits $x=a$, and $x=b$, and therefore is said to be continuous.

And $\Sigma_a^b f(x)$ denotes the summation of the series whose general term is $f(x)$, where x takes the values of every whole number between the limits $x=a$, and $x=b$, and is therefore discontinuous.

The theorems in articles 7, 8 and 9 are, I believe, entirely new. The first of these will be found useful in the summation of series by means of the decomposition of fractions, in consequence of showing the conditions that must exist amongst the constants to render the series one which may be summed, and thus entirely obviate the necessity of vinculating two terms, an expedient which was, I believe, first adopted by Mr. Woolhouse, and subsequently by Mr. Beecroft.

1. It is required to decompose the fraction $\frac{1}{(x-a)^n\,(x-b)^p\,(x-c)^q\,\ldots}$ into its simple factors.

Assume

$$\frac{1}{(x-a)^n\,(x-b)^p\,(x-c)^q\,\ldots} = \frac{A+B(x-a)+C(x-a)^2+\ldots\text{to } n \text{ terms,}}{(x-a)^n}$$
$$+\frac{A_1+B_1(x-b)+C_1(x-b)^2+\ldots\text{to } p \text{ terms,}}{(x-b)^p}$$
$$+\frac{A_2+B_2(x-c)+C_2(x-c)^2+\ldots\text{to } q \text{ terms,}}{(x-c)^q}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where A, B, C, *etc.*; A_1, B_1, C_1, *etc.*; A_2, B_2, C_2, *etc.*, are arbitrary constants which have to be determined.

That this assumption will satisfy the conditions of the question, appears

rom the circumstance, that there are $n+p+q+\ldots.$ arbitrary constants, nd $n+p+q+\ldots.$ terms, one of which is absolute, and the others ontain the powers of x, the coefficients of which may be made to fulfil $n+p+q+\ldots.-1$) conditions, and the absolute term another condition.

In the case before us we shall have to equate the absolute term with unity, nd the coefficients of the respective powers of x with zero.

These arbitrary constants may be determined by reducing the fractions to a common denominator, and collecting the coefficients of the powers of x in he following manner.

Put the absolute term $= F(A, A_1, etc.; B, B_1, etc. etc.)$; then the coefficients of x, x^2, x^3, *etc.* will be $F_1(A, A_1, etc.; B, B_2, etc. etc.)$; $F_2(A, A_1, etc.; B, B_1, etc. etc.); \ldots\ldots$ respectively: therefore we shall have

$$F(A, A_1, etc.; B, B_1, etc.\ldots) = 1$$
$$F_1(A, A_1, etc.; B, B_1, etc.\ldots) = 0$$
$$F_2(A, A_1, etc.; B, B_1, etc.\ldots) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

from which we may by the ordinary methods of elimination obtain the values of A, B, *etc*; $A_1, B_1, etc.\ldots\ldots\ldots$

This method would be exceedingly laborious in the decomposition of a fraction so general as the above, and thus render the determination of these arbitrary constants a matter of extreme difficulty; but their value may be readily found by means of the following artifice.

$$\text{Let } f(x) = \frac{1}{(x-b)^p\,(x-c)^q\,\ldots}; \quad f_1(x) = \frac{1}{(x-a)^n\,(x-c)^q\,\ldots};$$
$$f_2(x) = \frac{1}{(x-a)^n\,(x-b)^p\,\ldots}, \ etc.$$

then equation (1) by multiplying by $(x-a)^n$ will become

$$f(x) = A+B(x-a)+C(x-a)^2+D(x-a)^3+\ldots.$$
$$+\left\{\frac{A_1+B_1(x-b)+C_1(x-b)^2+D_1(x-b)^3+\ldots}{(x-b)^p}\right\}(x-a)^n$$
$$+\left\{\frac{A_2+B_2(x-c)+C_2(x-c)^2+D_2(x-c)^3+\ldots}{(x-c)^q}\right\}(x-a)^n$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now if we make n successive differentiations, and put $x=a$, every term which contains the factor $(x-a)$ will consequently vanish; therefore we shall have

$$f(a) = A; \frac{f'(a)}{1} = B; \ \frac{f''(a)}{1.2} = C; \frac{f'''(a)}{1.2.3} = D, etc.$$

$$\text{Similarly } f(b) = A_1; \frac{f'(b)}{1} = B_1; \ \frac{f''(b)}{1.2} = C_1; \frac{f'''(b)}{1.2.3} = D_1, etc.$$

$$f(c) = A_2; \frac{f'(c)}{1} = B_2; \ \frac{f''(c)}{1.2} = C_2; \frac{f'''(c)}{1.2.3} = D_2, etc.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Substituting these values in equation (1) we shall have

$$\frac{1}{(x-a)^n\,(x-b)^p\,(x-c)^q\,\ldots}=\frac{f(a)}{(x-a)^n}+\frac{f'(a)}{(x-a)^{n-1}}+\frac{f''(a)}{1.2(x-a)^{n-2}}$$

$$+\frac{f'''(a)}{1.2.3(x-a)^{n-3}}+\ldots\text{to } n \text{ terms},$$

$$+\frac{f(b)}{(x-b)^p}+\frac{f'(b)}{(x-b)^{p-1}}+\frac{f''(b)}{1.2(x-b)^{p-2}}$$

$$+\frac{f'''(b)}{1.2.3(x-b)^{p-3}}+\ldots\text{to } p \text{ terms},$$

$$+\frac{f(c)}{(x-c)^q}+\frac{f'(c)}{(x-c)^{q-1}}+\frac{f''(c)}{1.2(x-c)^{q-2}}$$

$$+\frac{f'''(c)}{1.2.3(x-c)^{q-3}}+\ldots\text{to } q \text{ terms},$$

$$+\textit{etc.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The above, which has been obtained from principles purely analytical, enables us to decompose rational fractions in a more simple manner than any other theorem that has come under my observation.

It will be necessary here to remark, that this theorem, as it now stands, is equally as general as when the numerator of the fraction, which has to be decomposed, contains a given function of x, which must be of lower dimensions than the denominator; for it will readily appear that only the functions $f(a)$, *etc.*, $f_1(b)$, *etc.*, will be affected by such a transformation.

The calculation of $f(a)$, $f'(a)$, *etc.*; $f_1(b)$, $f_1'(b)$, *etc.*, will of course become tedious for high values of n, p, q, *etc.*; in consequence of which, I shall subjoin the following table, which gives the values of all the functions to $f^{\text{v}}(a)$ in general terms.

2. Table for the values of the functions $f(a)$, $f'(a)$, *etc.*

$$f(a)=\frac{1}{(a-b)^p\,(a-c)^q\,(a-d)^r\,\ldots}$$

$$f'(a)=f(a)\times\left\{\frac{-p}{a-b}+\frac{-q}{a-c}+\ldots\right\}.$$

$$f''(a)=f(a)\times\left\{\left(\frac{p}{a-b}+\frac{q}{a-c}+\ldots\right)^2+\left(\frac{p}{(a-b)^2}+\frac{q}{(a-c)^2}+\ldots\right)\right\}.$$

$$f'''(a)=f(a)\times\left\{\left(\frac{-p}{a-b}+\frac{-q}{a-c}+\ldots\right)^3+3\left(\frac{-p}{a-b}+\frac{-q}{a-c}+\ldots\right).\right.$$

$$\left.\left(\frac{p}{(a-b)^2}+\frac{q}{(a-c)^2}+\ldots\right)+\left(\frac{-2p}{(a-b)^3}+\frac{-2q}{(a-c)^3}+\ldots\right)\right\}.$$

$$f^{\text{iv}}(a)=f(a)\times\left\{\left(\frac{p}{a-b}+\frac{q}{a-c}+\ldots\right)^4+6\left(\frac{p}{a-b}+\frac{q}{a-c}+\ldots\right)^2.\right.$$

$$\left(\frac{p}{(a-b)^2}+\frac{q}{(a-c)^2}+\ldots\right)+3\left(\frac{p}{(a-b)^2}+\frac{q}{(a-c)^2}+\ldots\right)^2$$

$$+4\left(\frac{-p}{a-b}+\frac{-q}{a-c}+\ldots\right).\left(\frac{-2p}{(a-b)^3}+\frac{-2q}{(a-c)^3}+\ldots\right)$$

$$\left.+\left(\frac{2.3.p}{(a-b)^4}+\frac{2.3.q}{(a-c)^4}+\ldots\right)\right\}.$$

$$f^{\text{v}}(a)=f(a)\times\left\{\left(\frac{-p}{a-b}+\frac{-q}{a-c}+..\right)^5+10\left(\frac{-p}{a-b}+\frac{-q}{a-c}+..\right)^3.\right.$$
$$\left(\frac{p}{(a-b)^2}+\frac{q}{(a-c)^2}+..\right)+15\left(\frac{-p}{a-b}+\frac{-q}{a-c}+..\right).$$
$$\left(\frac{p}{(a-b)^2}+\frac{q}{(a-c)^2}+..\right)^2+10\left(\frac{p}{a-b}+\frac{q}{a-c}+..\right)^2.$$
$$\left(\frac{-2p}{(a-b)^3}+\frac{-2q}{(a-c)^3}+..\right)+10\left(\frac{p}{(a-b)^2}+\frac{q}{(a-c)^2}+..\right).$$
$$\left(\frac{-2p}{(a-b)^3}+\frac{-2q}{(a-c)^3}+..\right)+5\left(\frac{-p}{a-b}+\frac{-q}{a-c}+..\right).$$
$$\left.\left(\frac{2.3.p}{(a-b)^4}+\frac{2.3.q}{(a-c)^4}+..\right)+\left(\frac{-2.3.4.p}{(a-b)^5}+\frac{-2.3.4.q}{(a-c)^5}+..\right)\right\}.$$

The series within ( ) must be taken to as many terms as there are factors in $f(a)$.

From this table we shall readily get the values of $f_1(b)$, $f_1'(b)$, *etc.*; $f_2(c)$, $f_2'(c)$, *etc.*; for we have only to substitute b for a, and a for b, *etc.*; which will give us the values of the functions. It will be unnecessary for me to enter any further into the details of finding these functions, since those who are likely to use them will readily see how they are to be obtained.

These functions, however, might have been much simplified, but my object has been to present them in such a form, that they may be easily extended to any number of factors.

In order, therefore, to exemplify the theorem (2), and show the facility with which we shall be enabled to decompose fractions, we shall proceed to the complete decomposition of some of the most remarkable of those which have been previously decomposed.

(3). If in theorem (2) we take $n=p=q=..=1$, we shall obtain the following result:—

$$\frac{1}{(x-a)(x-b)(x-c)..}=\frac{f(a)}{(x-a)}+\frac{f_1(b)}{(x-b)}+\frac{f_2(c)}{(x-c)}+..,$$ which is the ordinary theorem that is given for decomposing fractions of this nature. (See Murphy on the Theory of Equations, p. 97.)

Hence the integral $\int\frac{1}{(x-a).(x-b).(x-c)\,....}\,.\,dx$, may be readily obtained; and also $\frac{d^m}{dx^m}\left(\frac{1}{(x-a).(x-b).(x-c)\,....}\right)$ fully determined.

Now, let us take a, b, c, *etc.* in arithmetical progression, that is, put $-a=a$, $-b=2a$, $-c=3a$, *etc.*; then the above equation will become

$$\frac{1}{(x+a)(x+2a)(x+3a)....}=\frac{f(-a)}{x+a}+\frac{f_1(-2a)'}{x+2a}+\frac{f_2(-3a)}{x+3a}+....$$

It will readily be seen that $f(-a)=-f_{m-1}(^{-ma})$; $-f_1(-2a)=f_{m-2}(^{-(m-1)a})$; $f_2(-3a)=-f_{m-3}(^{-(m-2)a})$, *etc.*, where m is the number of factors in the denominator; consequently, when m is even, the above equation may be put in the following form:

$$\frac{1}{(x+a)(x+2a)(x+3a)\ldots\ldots}=f(-a)\left\{\frac{1}{x+a}-\frac{1}{x+ma}\right\}-$$
$$f_1(-2a)\left\{\frac{1}{x+2a}-\frac{1}{x+(m-1)a}\right\}+\textit{etc. to } \frac{m}{2} \text{ terms,}$$

which will enable us to sum the series to infinity, whose general term is $\frac{1}{(x+a)(x+2a)\ \ldots\ldots}$, where x takes the successive values 1, 2, 3, *etc.*, ad infinitum.

Put $\Sigma_a^m \mathrm{F}(x)= \mathrm{F}(a)+\mathrm{F}(a+1)+\mathrm{F}(a+2)+\ldots$ to m terms. Then agreeably to this notation we shall have, from the same considerations employed by Mr. Woolhouse, (see Ladies' Diary, 1836, p. 63)

$$\Sigma_1^\infty\left(\frac{1}{(x+a)(x+2a)\ldots(x+ma)}\right)=f(-a)\left\{\Sigma_1^{ma}\left(\frac{1}{x+a}\right)-\Sigma_1^{a}\left(\frac{1}{x+ma}\right)\right\}$$
$$+f_1(-2a)\left\{\Sigma_1^{(m-1)a}\left(\frac{1}{x+2a}\right)-\Sigma_1^{2a}\left(\frac{1}{x+(m-1)a}\right)\right\}$$
$$+f_2(-3a)\left\{\Sigma_1^{(m-2)a}\left(\frac{1}{x+3a}\right)-\Sigma_1^{3a}\left(\frac{1}{x+(m-2)a}\right)\right\}$$
$$+\textit{ etc., to } \frac{m}{2} \text{ terms} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

And similarly, when m is odd, we shall have

$$\Sigma_1^\infty\left(\frac{1}{(x+a)(x+2a)\ldots(x+ma)}\right)=f(-a)\left\{\Sigma_1^{ma}\left(\frac{1}{x+a}\right)+\Sigma_1^{a}\left(\frac{1}{x+ma}\right)\right\}$$
$$+f_1(-2a)\left\{\Sigma_1^{(m-1)a}\left(\frac{1}{x+2a}\right)+\Sigma_1^{2a}\left(\frac{1}{x+(m-1)a}\right)\right\}$$
$$+f_2(-3a)\left\{\Sigma_1^{(m-2)a}\left(\frac{1}{x+3a}\right)+\Sigma_1^{3a}\left(\frac{1}{x+(m-2)a}\right)\right\}$$
$$+\textit{ etc., to } \frac{m-1}{2} \text{ terms.}$$

$$+f_{\frac{m-1}{2}}\left(-\tfrac{m+1}{2}.a\right)\left\{\Sigma_1^{(m+1)a}\left(\frac{1}{x+\frac{m+1}{2}.a}\right)-\Sigma_1^{\frac{m+1}{2}.a}\left(\frac{1}{x+(m-1)a}\right)\right\} \ldots (4)$$

The above notation for summation is analogous to that of definite integrals; and may possibly lead us to discover some remarkable connection between these interesting subjects.

To show the ease with which series of the above nature can be summed, I shall select one example containing several factors.

Required the sum of the series

$$\frac{1}{2.3.4.5.6.7.8.9}+\frac{1}{3.4.5.6.7.8.9.10}+\frac{1}{4.5.6.7.8.9.10.11}+\ldots \text{ ad infin.}$$

Here $a=1$; $m=8$; $f(-a)=\frac{1}{1.2.3.4.5.6.7}$; $f_1(-2a)=-\frac{1}{1.2.3.4.5.6}$;

$$f_2(-3a)=\frac{1}{2.1.2.3.4.5};\ f_3(-4a)=-\frac{1}{3.2.1.1.2.3.4};$$

$$\Sigma_1^8\left(\frac{1}{x+1}\right)-\Sigma_1^1\left(\frac{1}{x+8}\right)=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{9}=\frac{481}{2.4.5.7}$$

$$\Sigma_1^7\left(\frac{1}{x+2}\right)-\Sigma_1^2\left(\frac{1}{x+7}\right)=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{8}-\frac{1}{9}=\frac{153}{4.5.7}$$

$$\Sigma_1^6\left(\frac{1}{x+3}\right)-\Sigma_1^3\left(\frac{1}{x+6}\right)=\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{7}-\frac{1}{8}-\frac{1}{9}=\frac{74}{4.5.6}$$

$$\Sigma_1^5\left(\frac{1}{x+4}\right)-\Sigma_1^4\left(\frac{1}{x+5}\right)=\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{6}-\frac{1}{7}-\frac{1}{8}-\frac{1}{9}=\frac{1}{5}.$$

Then from equation (3) we have

$$\frac{1}{2.3.4..9}+\frac{1}{3.4.5..10}+\frac{1}{4.5.6..11}+\text{ad infin.}=\frac{481}{2^2.3.4^2.5^2.7.^26}-\frac{153}{2.3.4^2.5^2.6.7}$$

$$+\frac{74}{2^2.3.4^2.5^2.6}-\frac{1}{2^2.3^2.4.5}=\frac{1}{1.2^2.3.4^2.5.6.7^2}.$$

I have in this example set down the work at full length, in order to show how readily these series can be summed by means of equations (3) and (4). The result may however be verified mentally in a few minutes.

The binomials in equations (3) and (4) may be reduced to single terms but the equations would then be less symmetrical; and besides, no advantage in the arithmetical computation would arise from such a reduction. Therefore I feel justified in allowing the equations to remain in their present form.

(To be continued.)

MATHEMATICAL EXERCISES—*(continued.)*

33.—*By* $\pm$.

Let three parabolas be escribed to the three sides of a triangle, aud have their principal axes in the lines bisecting the exterior angles: then if a, b, c represent the sides of the triangle, s their half sum, and p_1, p_2, p_3 the semi-parameters of the three parabolas, it is required to show that

$$p_1p_2p_3=\frac{(a-b)(b-c)(c-a)s^3}{abc}.$$

34.—*Mr. Thomas Weddle, Newcastle.*

Sum each of the series,

$$(\sec\theta+1)(\sec\frac{\theta}{2}+1)(\sec\frac{\theta}{2^2}+1)\ldots.(\sec\frac{\theta}{2^{n-1}}+1),$$

$$(2\cos\theta+1)(2\cos\frac{\theta}{3}+1)(2\cos\frac{\theta}{3^2}+1)\ldots.(2\cos\frac{\theta}{3^{n-1}}+1),$$

$$(2\cos\theta-1)(2\cos\frac{\theta}{3}-1)(2\cos\frac{\theta}{3^2}-1)\ldots.(2\cos\frac{\theta}{3^{n-1}}-1).$$

35.—*Mr. Matthew Collins, Limerick.*

How many terms of the squares of the numbers 1, 2, 3, 4, *etc.*, must be added that the sum may be a rational square number?

36.—*Mr. J. W. Elliott, Greatham.*

Find the equation of a plane meeting three given straight lines in space, and making equal angles with them.

37.—*By ϕ.*

The six vertices of two triangles about a conic section lie also in a conic section.

38.—*By W. F., Durham.*

(1.) If from any point P in the plane of a plane quadrilateral ABCD, lines be drawn to the four angular points, then (the diagonals being drawn) there exists the following relation between the triangles,

$$\text{APC.PBD}=\text{APD.BPC} \pm \text{PDC.PAB},$$

the upper or lower sign being taken according as the point P is *without* or *within* the quadrilateral.

(2.) If from both extremities of any line PQ in space, lines be drawn to the four angular points of a quadrilateral ABCD in space, then (the diagonals being drawn) there exists the following relation between the pyramids,

$$\text{APQC.PQDB}=\text{APQD.BCQP} \pm \text{PQDC.PQAB},$$

the upper or lower sign being taken according as the line PQ is *without* or *within* the quadrilateral.

39.—*Mr. W. S. B. Woolhouse, Editor of the Lady's and Gentleman's Diary.*

In laying down *fifteen* points upon a plane, they may be so placed that sets of three points, ranging exactly in a line, shall exist in *twenty-two* different directions. Required proof.

*** This exercise was proposed in the last No. of the Northumbrian Mirror; but as that work has been discontinued, it is re-proposed here at the request of the author.

40.—*Mr. Fenwick.*

Prove that if the three lines,

$$\left.\begin{array}{l} y=a_1z+a' \\ x=\beta_1z+\beta' \end{array}\right\}\cdot\cdot(1) \qquad \left.\begin{array}{l} y=a_2z+a'' \\ x=\beta_2z+\beta'' \end{array}\right\}\cdot\cdot(2) \qquad \left.\begin{array}{l} y=a_3z+a''' \\ x=\beta_3z+\beta''' \end{array}\right\}\cdot\cdot(3)$$

be mutually conjugate to one another (the perpendicular case included), there exists the relation

$$\frac{1}{a_1\beta_3}+\frac{1}{a_2\beta_1}+\frac{1}{a_3\beta_2}=\frac{1}{\beta_1a_3}+\frac{1}{\beta_2a_1}+\frac{1}{\beta_3a_2}.$$

41.—*By Pen-and-Ink.*

Give *practicable* constructions of the two following important problems:—

(1.) Given the elevations of two lines (which intersect) above the horizon, and the angle which they form with each other; to find the angle formed by their orthographic projections on the plane of the horizon.

(2.) Given the depressions of two lines below the horizon, and their azimuths, to find the inclination of the plane which contains them to the horizon, and likewise the azimuth of the line of greatest inclination to the horizon, which can be drawn in that plane.

42.—*Mr. Fenwick.*

If the opposite faces of a hexahedron inscribed in a surface of the second order be produced to meet, the three lines of intersection will be in the same plane: Also, if an octahedron circumscribe the same surface (the angular points of the former being the points of contact of the latter), the three lines which join its opposite angular points, two and two, will pass through the same point.

SOLUTIONS OF MATHEMATICAL EXERCISES.

XI.—*Mr. Thomas Dobson*

A, B, C are three given points in a straight line, of which C is the centre of a given circle, and a straight line is drawn through B intersecting the circle in the points D, E: find the position of this line when the chord D E appears a maximum to a spectator at A.

[Solution.—*Mr. James Anderson.*]

The student will readily sketch the figure.

The points A and B may be on the same side of C, or they may be on different sides: B may be nearer C than A is, or it may be more distant: and B and A may be both, or either, at a greater or less distance from C than the radius of the circle. The following solution is adapted to the hypothesis, that A and B are on the same side of C, both exterior to the circle, and A more distant than B. The result, is applicable to all the cases, CB being considered negative when A and B are on opposite sides.

Draw CH perpendicular to DE, and let $CA = a$, $CB = b$, the radius of the circle $= r$, and the angle $CBE = \theta$. Then $BH = b \cos\theta$, $CH = b \sin\theta$, $DH = \sqrt{r^2 - b^2 \sin^2\theta}$, $BD = BH - DH$, $BE = BH + DH$, $AD^2 = AB^2 + BD^2 + 2\, AB \cdot BD \cos\theta$, and $AE^2 = AB^2 + BE^2 + 2\, AB \cdot BE \cos\theta$.

Also,
$$\sec^2 DAE = \frac{4AD^2 . AE^2}{(AD^2 + AE^2 - DE^2)^2}.$$

Substituting in the second member of this equation the preceding values of its terms, we find without difficulty,

$$\sec^2 DAE = \frac{4\{(a-b)^2 + r^2 - b^2 + 2ab\cos^2\theta\}^2 - 16a^2\cos^2\theta(r^2 - b^2\sin^2\theta)}{4\{(a-b)^2 - r^2 + b^2 + 2b(a-b)\cos^2\theta\}^2}$$

$$= \frac{(a^2 - r^2)^2 - 4a(a-b)(ab - r^2)\sin^2\theta}{\{a^2 - r^2 - 2b(a-b)\sin^2\theta\}^2}.$$

By the question the angle DAE is to be a maximum; now it is evident that if it be not greater than $\frac{\pi}{2}$, it will be a maximum when the square of its secant is a maximum; but DAE will not be greater than $\frac{\pi}{2}$, unless A fall within the circle; that is, unless $a < r$, a case which we need not consider, as it is obvious that the maximum value of DAE is then π. To find the maximum and minimum values of DAE, it is, therefore, merely necessary to differentiate $\sec^2 DAE$ with respect to θ, and equate the result to zero. This gives

$$\frac{8(a-b)^2 \sin\theta\cos\theta\{r^2(a-b)(a^2 - r^2) - 2ab(a-b)(ab - r^2)\sin^2\theta\}}{\{a^2 - r^2 - 2b(a-b)\sin^2\theta\}^3} = 0.$$

Hence we have either

$$\theta = 0, \text{ or } \theta = \frac{\pi}{2}, \text{ or } \sin^2\theta = \frac{r^2(a^2 - r^2)}{2ab(ab - r^2)}.$$

For $\theta = 0$, the value of DAE is 0, and a minimum, unless A be within the circle, when DAE is a maximum and equal to π. For $\theta = \frac{\pi}{2}$, the value of sec DAE becomes $1 + \frac{2(r^2 - b^2)}{a^2 - r^2 - 2b(a-b)}$, which is an impossible result,

unless $b^2 < r^2$. Hence, as is indeed evident, $\theta = \frac{\pi}{2}$ does not correspond to either a maximum or minimum value, or any real value, unless B fall within the circle. When b lies between the positive value $\frac{r^2+r\sqrt{2a^2-r^2}}{2a}$, and the negative value $\frac{r^2-r\sqrt{2a^2-r^2}}{2a}$, $\theta = \frac{\pi}{2}$, corresponds to a maximum, and that maximum is determined by the condition $\cos^2\frac{1}{2}\text{DAE} = \frac{(a-b)^2}{a^2+r^2-2ab}$; when B lies within the circle, and, this condition being fulfilled, b has any value which does not lie between the limits last specified, $\theta = \frac{\pi}{2}$ corresponds to a minimum, but not an *absolute* minimum. When b has a value between $\frac{r^2+r\sqrt{2a^2-r^2}}{2a}$ and $\frac{r^2-r\sqrt{2a^2-r^2}}{2a}$, that is, when $\theta = \frac{\pi}{2}$ gives the maximum, the equation $\sin^2\theta = \frac{r^2(a^2-r^2)}{2ab(ab-r^2)}$ becomes impossible, as it would imply that $\sin^2\theta > 1$. For all other values of b, this equation gives the value of θ corresponding to a maximum value of DAE, and this value will be found from the equation

$$\sec^2\text{DAE} = \frac{a^2(ab-r^2)^2}{b(a^2-r^2)(a^2b+br^2-2ar^2)}.$$

Let us take the case $b = r$, or let D and B coincide; then from

$$\sin^2\theta = \frac{r^2(a^2-r^2)}{2ab(ab-r^2)}$$

we find $\sin^2\theta = \frac{a+r}{2a}$; but $\cos^2 2\theta = 1 - 2\sin^2\theta$; hence

$$\cos 2\theta = -\frac{r}{a}, \text{ or } \cos(\pi - 2\theta) = \frac{r}{a}.$$

Also in this case $\sec^2\text{DAE} = \frac{a^2}{a^2-r^2}$, or $\cos^2\text{DAE} = \frac{a^2-r^2}{a^2}$, and $\sin\text{DAE} = \frac{r}{a}$. Hence $\text{DAE} = 2\theta - \frac{\pi}{2}$, or the place of E is determined by the intersection of the semicircle, described upon AC, with the given circle,—a result easily arrived at from elementary geometry.

If $a = r$, and $b > r$, the formula gives $\text{DAE} = \frac{\pi}{2}$, and this is true, however little b may exceed r. If $a = r$, and $b < r$, the maximum is π, however little b may fall short of r.

When a becomes infinite, $\sin^2\theta = \frac{r^2}{2b^2}$, whilst b has such a value as to render this value of $\sin^2\theta$ possible. When $a = \infty$, and $2b^2 = r^2$, then $\theta = \frac{\pi}{2}$. Of course DAE, in this case, vanishes.

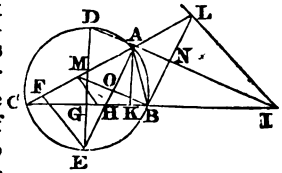
A good solution was also received from the Proposer.

XXV.—*Mr. Matthew Collins, Limerick.*

Given the bisectors of the interior and exterior angles at the vertex, and the sum or difference of the sides, to construct the plane triangle.

[First Solution.—*Mr. M. Collins, the proposer.*]

Analysis.—Let ABC be the required triangle, and DE the diameter of the circumscribed circle, cutting the base BC at right angles in G. Draw AE, AD, and produce DA to meet CB produced in I, and let AE intersect CB in H; then AH, AI are the bisectors of the interior and exterior angles at the vertex, and if EF be drawn perpendicular to AC, the segment AF will be equal to half the sum of the sides, and CF will be equal to half their difference. Draw also AK perpendicular to BC; then the right-angled triangle HAI being given, the segments of the base HK, KI, as well as the perpendicular AK, are all given; but (*Student, Modern Geometry*, Prop. 34 and cor.) $GK.GI = AF^2$ and $GK.GH = CF^2$; and when the sum of the sides is given, AF is given; and since IK is a given line, and IG.GK a given rectangle, the point of section G is given. Also when the difference of the sides is given, CF is given; and because HK is a given line, and KG.GH a given rectangle, the point of section G is, in this case also, given. This analysis suggests the following method of construction.



Construction.—Place the interior and exterior bisectors AH, AI at right angles to each other; join HI and draw AK perpendicular thereto. Then if the sum of the sides be given, produce IK to G so that the rectangle IG.GK may be equal to the square of half the sum of the sides, and if the difference of the sides be given, produce KH to G so that the rectangle HG.GK may be equal to the square of half the difference of the sides. Through G draw EGD at right angles to IHG meeting AH and IA produced in E and D. On ED as a diameter describe a circle cutting IH produced in B and C, and passing through A since the angle DAE is a right angle; draw AC, BC and ABC is the required triangle, as is evident from the analysis.

[Second Solution.—*Mr. John Laws, Newcastle-on-Tyne.*]

Construction.—With the bisectors of the interior and exterior vertical angles, construct the right-angled triangle AHI (*preceding figure*); draw IL, HM making angle AIL = AIH, and angle AHM = AHI; through the point A draw (Leslie's Geom. Anal. Prop. 29, p. 99) the straight line CL, limited by the lines IL and IH produced, equal to the given *sum* of the sides of the triangle; take IB = IL and draw AB, then ABC is the triangle required. But if the difference of the sides be one of the data, through A draw (Leslie's Geom. Anal. Prop. 30, p. 101) AC so that the segment MC, intercepted between the lines HM and IH produced, shall be equal to the given *difference* of the sides; take HB = HM, and draw AB, then ABC will be the triangle sought.

Demonstration.—Draw BL, BM cutting AI, AH in N and O respectively; then, by construction, the triangles BIL, BHM are isosceles, and IN, HO bisect the angles BIL, BHM; therefore IN, HO bisect BL, BM at right angles in N and O; and hence it is obvious that AL = AB, and AB = AM; consequently AH, AI are the bisectors of the interior and exterior vertical angles, and CA + AB = CL, the *sum* of the sides, and AC—AB = CM, the *difference* of the sides.

[THIRD SOLUTION.—*Mr. Hugh Godfray, Jersey.*]

Let ABC be the triangle required (*same figure*), AH, AI the interior and exterior bisectors of the angles at the vertex; then the right-angled triangle HAI is given. Draw IL making the angle AIL = AIC and meeting CA produced in L; then the triangles ABI and AIL are obviously equiangular, and the side AI is common to both; therefore AL = AB and IL = IB; hence there are given the side CL, the opposite angle CIL, and the length of the line IA bisecting the given angle CIL, to construct the triangle CLI. The proposition is therefore reduced to Prob. LXXII, Simpson's Algebra, Appendix, p. 388.

When the difference of the sides is given, draw HM making the angle AHM = AHB, meeting CA in M; then the triangle AMH is evidently equal to the triangle ABH; hence AM = AB, HM = HB, and therefore CM = CA—AB. Consequently in the triangle CMH there are given the base CM, the angle CHM, and the line HA bisecting the *exterior* angle MHB to construct it, a problem very similar to the one to which the former case has been referred.

Mr. Thomas Weddle favoured us with a good solution to this exercise.

XXVI.—*Mr. G. W. Hearn, Royal Military College, Sandhurst.*

If $a, b, c, \ldots. k$ be any n quantities; then will

$$\frac{abc..}{a(a+b)(a+b+c)..}+\frac{bac..}{b(b+c)(b+a+c)..}+\frac{cab..}{c(c+a)(c+a+b)..}+etc.=1.$$

[SOLUTION.—*Mr. Hugh Godfray, Jersey, and the proposer, whose solutions were almost identical.*]

The numerators of the fractions are the permutations of the n factors $a, b, c \ldots. k$. The number of fractions is therefore $n(n-1)(n-2)\ldots 3.2.1$; and they can evidently be arranged in n different sets according to the last letter in each, the number in each set being $(n-1)(n-2)\ldots. 3.2.1$:—thus we shall have one set ending with a, another with b, and so on. Moreover, it is easily seen that the last factor of the denominator (which, being the sum of n quantities, is the same in all the quantities) is the only factor containing the last letter of the numerator.

Now, if in any one of the sets, that ending with a for instance, we leave out the last factor of both numerator and denominator in each fraction, then the sum of the $(n-1)(n-2)\ldots.3.2.1$ resulting fractions will be the same function of the $(n-1)$ remaining letters $b, c, \ldots. k$ that the given function is of all the n letters. Hence, representing by F_n the given function, we have

$$F^n\ (a,b,c..k) =$$
$$\frac{a}{a+b+c+..k}.\{F_{n-1}(b,c..k)\}+\frac{b}{a+b+c+..k}.\ \{F_{n-1}(a,c..k)\}+\ldots.$$

If, therefore, we suppose the theorem to be true for $(n-1)$ quantities, each of the functions

$$F_{n-1}(b,c..k),\qquad F_{n-1}(a,c..k),\ etc.,$$

becomes equal to *unity*, and we get

$$F_n\ (a,b,c..k)=\frac{a}{a+b+c+..k}+\frac{b}{a+b+c+..k}+\ldots.$$
$$=\frac{a+b+c+..k}{a+b+c+..k}=1;$$

that is, it will also be true for n quantities. But it is easily shewn to be true for *two* factors, for then we have

$$\frac{ab}{a(a+b)}+\frac{ba}{b(b+a)}=\frac{b}{a+b}+\frac{a}{a+b}=\frac{a+b}{a+b}=1.$$

Therefore by induction the theorem is universally true.

XXVII.—*C. F. B.*

Into a cubical cistern, eight feet deep, and having an unknown leak, water is poured from two pumps, worked by two men A and B. They pump together till the vessel is half filled, when B falls asleep. A continues pumping till it is $\frac{3}{4}$ filled, and then goes away. B afterwards waking finds the cistern still half full, and after pumping till it is again $\frac{3}{4}$ filled, departs also, and meeting with A charges him with leaving his work unfinished. They return together, and find the water $1\frac{1}{2}$ inches lower than when B left. The leak is now discovered and stopped: and by their joint efforts the vessel is filled in half the time they had worked together at first. They remark also that $10\frac{1}{3}$ hours had elapsed since they first began pumping, and that B had worked alone twice as long as A had. Supposing that a cubic foot contains $15\frac{5}{8}$ gallons, required the quantity of water thrown in by each pump, as well as the quantity discharged at the leak, *while one or both were pumping.*

[First Solution.—*By the Proposer.*]

Let x denote the time (in hours) in which A and B work jointly *after* the leak is stopped; y the time which A works alone: then $2x$ is the time which A and B work together *originally*, and $2y$ is the time which B works alone.

It is easily computed that the cistern contains 8000 gallons; and that the part to be filled by A and B jointly after the leak is stopped, is 2125 gallons, which is filled in x hours. Hence they originally pump 4250 gallons in $2x$ hours; and since the cistern is then only half full, the leak will have discharged 250 gallons in $2x$ hours. Wherefore to find the quantities discharged while each worked alone, viz. in y hours and $2y$ hours, we have respectively,

$$2x : y :: 250 : \frac{125y}{x}, \text{ and } 2x : 2y :: 250 : \frac{250y}{x};$$

and therefore A pumps $2000+\frac{125y}{x}$ in y hours, and B pumps $2000+\frac{250y}{x}$ in $2y$ hours; and together they would pump $3000+\frac{250y}{x}$ in y hours. Whence we have the following analogy:—

$$y : x :: 3000+\frac{250y}{x} : 2125; \text{ or } y=\frac{8x}{5}.$$

To obtain the times in which the leak was discharging while *neither* A *nor* B *was working*, we have the following proportions:—

$$250 : 2000 : 2x : 16x, \text{ and } 250 : 125 :: 2x : x$$

The six intervals, therefore, distinguished in the question, are as follows:—

A and B work together $=2x$	B alone $=2y=\frac{16x}{5}$
A alone $=y=\frac{8x}{5}$	Second discharge of leak $=x$
First discharge of leak $=16x$	A and B together finally $=x$

Hence we have by the question

$$2x + \frac{8x}{5} + 16x + \frac{16x}{5} + x + x = \frac{31}{3}; \text{ or } x = \frac{5}{12}.$$

Also $\qquad y = \frac{8x}{5} = \frac{2}{3}$; and $\frac{y}{x} = \frac{8}{5}$.

Whence A works *alone or jointly*, $\frac{5}{6}, \frac{2}{3}$ and $\frac{5}{12}$ hours;

and B .... $\frac{5}{6}, \frac{4}{3}$ and $\frac{5}{12}$ ......

But by what has preceded,

A pumps $2000 + \frac{125 \times 8}{5} = 2200$ gallons in $\frac{2}{3}$ of an hour, and

B...... $2000 + \frac{250 \times 8}{5} = 2400$ .... $\frac{4}{3}$..............

and therefore the quantities they respectively pumped are thus found:—

$\frac{2}{3} : \frac{5}{6} + \frac{2}{3} + \frac{5}{12} :: 2200 = 6325$ gallons pumped by A;

$\frac{4}{3} : \frac{5}{6} + \frac{4}{3} + \frac{5}{12} :: 2400 = 4650$ ....B.

The times of the discharge of the leak, *while* A *or* B, *or both, were pumping*, were $\frac{5}{6}, \frac{2}{3}$, and $\frac{4}{3}$ respectively; and since it discharged 250 gallons in $\frac{5}{6}$ of an hour, we have

$\frac{5}{6} : \frac{5}{6} + \frac{2}{3} + \frac{4}{3} :: 250 : 850$ gallons discharged by the leak.

[SECOND SOLUTION.—*Mr. H. Godfray, Jersey.*]

Let the gallon be taken for unity; and let x, y, z be the quantities of water which flow from the pumps A and B, and from the leak. Then

$\frac{4000}{x+y-z}$ = time till B falls asleep.	$\frac{2000}{y-z}$ = time till B goes away.
$\frac{2000}{x-z}$ = ... A goes away.	$\frac{125}{z}$ = ... A and B return
$\frac{2000}{z}$ = ... B wakes.	$\frac{2125}{x+y}$ = ... the cistern is filled

Whence from the conditions of the question, we get the following equations:—

$$\frac{2125}{x+y} = \frac{1}{2} \cdot \frac{4000}{x+y-z}, \text{ or } x = 17z - y \text{ ....(1)}$$

$$\frac{2000}{y-z} = 2 \cdot \frac{2000}{x-z}, \text{ or } x = 2y - z \text{ ....(2)}$$

From these equations we have $x = 11z$ and $y = 6z$.

Again, the whole time elapsed is $10\frac{1}{3}$ hours; and therefore putting for x and y their values in terms of z, in the preceding expressions for the times, we get

$$\frac{4000}{16z} + \frac{2000}{10z} + \frac{2000}{z} + \frac{2000}{5z} + \frac{125}{z} + \frac{2125}{17z} = \frac{31}{3}.$$

Wherefore $z = 300$, $x = 3300$, $y = 1800$.

If now we substitute these values in the expressions for the times, and tabulate the results, we have

	h.	m.	during which time	A pumps	B pumps	leak.
Till B falls asleep	0	50		2750	1500	250
....A goes away	0	40		2200	——	200
....B wakes	6	40		——	——	2000
....B goes away	1	20		——	2400	400
....A and B return	0	25		——	——	125
....cistern is filled	0	25		1375	755	——
Total............	10	20		6325	4650	2975.

XXVIII.—*Mr. Thomas Dobson, Totteridge, Herts.*

From a point P in the plane of a given quadrilateral figure ABCD, right lines PQ, PR, PS, PT, are drawn parallel to lines given in position, to meet AB, BC, CD, AD, respectively. Determine the locus of P, when PR.PT : PS. PQ in a constant ratio.

[Solution.—*Mr. Weddle, Newcastle.*]

Join BD (the fig. is readily sketched) which bisect in O; take O for the origin of rectangular co-ordinates, OD being the axis of x. Let a_1, a_2, a_3, a_4 be the inclinations of AB, BC, CD and DA to the lines given in position or to PQ, PR, PS and PT respectively; also denote the perpendiculars from P upon AB, BC, CD and DA by p_1, p_2, p_3 and p_4; $\therefore$ PQ $= p_1$ cosec a_1, PR $= p_2$ cosec a_2, PS $= p_3$ cosec a_3 and PT $= p_4$ cosec a_4; hence PR.PT : PS.PQ : : $p_2\ p_4$ cosec a_2 cosec a_4 : $p_1\ p_3$ cosec a_1 cosec a_3 = a constant ratio, consequently, $p_2 p_4 : p_1 p_3$ = constant ratio $= 1 : m$ (suppose) $\therefore$ $p_1 p_3 = m\, p_2 p_4$.

Moreover if BO $=$ OD $= a$, and β_1, β_2, β_3, and β_4 be the angles ABD, CBD, CDB and ADB respectively, the equations of AB, BC, CD and DA will be as follows:

$$\text{Equation of AB is } \cos\beta_1\, y + \sin\beta_1\, (x+a) = 0,$$
$$\text{........ BC.... } \cos\beta_2\, y - \sin\beta_2\, (x+a) = 0,$$
$$\text{........ CD.... } \cos\beta_3\, y + \sin\beta_3\, (x-a) = 0,$$
$$\text{........ DA.... } \cos\beta_4\, y - \sin\beta_4\, (x-a) = 0.$$

Hence, x and y being the co-ordinates of P,

$$p_1 = \cos\beta_1\, y + \sin\beta_1\, (x+a), \qquad p_2 = \cos\beta_2\, y - \sin\beta_2\, (x+a),$$
$$p_3 = \cos\beta_3\, y + \sin\beta_3\, (x-a), \qquad p_4 = \cos\beta_4\, y - \sin\beta_4\, (x-a)$$

But $p_1 p_3 = m\, p_2 p_4$; hence,

$$\{\cos\beta_1.y + \sin\beta_1(x+a)\}.\{\cos\beta_3.y + \sin\beta_3(x-a)\}$$
$$= m.\{\cos\beta_2.y - \sin\beta_2(x+a)\}.\{\cos\beta_4.y - \sin\beta_4(x-a)\}.$$

This being multiplied out and reduced gives

$$\{\cos\beta_1\cos\beta_3 - m\cos\beta_2\cos\beta_4\}y^2 + \{\sin\beta_1\sin\beta_3 - m\sin\beta_2\sin\beta_4\}.(x^2 - a^2)$$
$$+\{\sin(\beta_1+\beta_3)+m\sin(\beta_2+\beta_4)\}xy + \{\sin(\beta_1-\beta_3)+m\sin(\beta_2-\beta_4)\}ay = 0.$$

This is the equation of the locus, and, as it is of the second degree, it shows that the curve is one of the conic sections, or one of their varieties.

Mr. Dobson, the proposer, also favoured us with a solution.

A solution of exercise 29 is given at page 240, by the proposer.

XXX.—*Mr. Rutherford.*

Let ABC be a plane triangle, and P any point in its plane; then if a, b, c denote the sides of the triangle respectively opposite to A, B, C; α, β, γ the respective distances of P from the same angular points, and Δ the area of a triangle whose three sides are $\alpha \sin A$, $\beta \sin B$, $\gamma \sin C$; it is required to prove that

$$2a^2 \text{cosec A} \sin B \sin C = \alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C \pm 8\Delta;$$

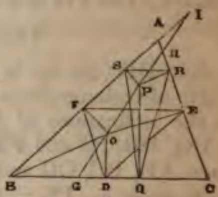
the sign + applying when the point P is *within*, and the sign — when the point is *without* the circumscribed circle.

[SOLUTION.—*Mr. Thomas Weddle; and similarly by Mr. S. Bills, Hawton.*]

Draw PQ, PR, PS perpendicular to BC, AC, and AB respectively, and conceive the lines AP, BP, CP to be drawn; then we have

$$AR = \alpha\cos RAP,\ AS = \alpha\cos PAS,\ PR = \alpha\sin RAP,\ PS = \alpha\sin PAS;$$

$$\therefore AR.AS - PR.PS = \alpha^2(\cos RAP\cos PAS - \sin RAP\sin PAS)$$
$$= \alpha^2\cos RAS = \alpha^2\cos A;$$



$\therefore$ 4 area ARPS—8 area PRS = 4 area ARS—
4 area PRS = 2AR.AS sinA—2PR.PS sinRPS
$= 2(AR.AS - PR.PS)\sin A = 2\alpha^2\cos A\sin A = \alpha^2\sin 2A.$

$\therefore$ 4 area ARPS $= \alpha^2 \sin 2A + 8$ area PRS.

Similarly, 4 area BQPS $= \beta^2 \sin 2B + 8$ area PQS,

and, 4 area CQPR $= \gamma^2 \sin 2C + 8$ area PQR.

Hence, by addition,

$$4 \text{ area ABC} = \alpha^2 \sin 2A + \beta^2 \sin B2 + \gamma^2 \sin 2C + 8 \text{ area QRS}.$$

Again, since a circle may be described through the points A, R, P, S, we have (Euc. vi. D.) AP.RS = AR.PS + AS.PR; therefore

$$\alpha.\ RS = \alpha^2 (\cos RAP \sin PAS + \sin RAP \cos PAS) = \alpha^2 \sin A;$$

hence RS $= \alpha \sin A$, and similarly QS $= \beta \sin B$, and QR $= \gamma \sin C$. Consequently the triangle QRS is that represented by Δ; also since 2 area ABC $= bc \sin A = a^2$ cosec A sin B sin C; hence

$$2\,a^2 \text{ cosec A} \sin B \sin C = \alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C + 8\Delta.$$

In a similar manner it may be shown when the point is without the circle described about the triangle ABC, that

$$2\,a^2 \text{ cosec A} \sin B \sin C = \alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C - 8\Delta.$$

Cor.—If P were in the circumference of the circle circumscribing the triangle ABC, RS would (Mathematician, p. 73) pass through Q, and therefore RS = QS + QR, but it has been shown that RS = PA sin A, BS = BP sin B, and QR = CP sin C; hence AP sin A = BP sin B + CP sin C; wherefore also AP·BC = BP·AC + CF·AB.

An excellent solution was also received from Mr. John Laws, Newcastle.

POLES AND POLARS IN SPACE.

[*Mr. Fenwick.*]

(Continued from page 242.)

18. *The poles of a plane in reference to all surfaces of the second degree which touch eight fixed planes, lie in one and the same straight line.*

For in this case, as in *art.* 16, one of the coefficients a, b, *etc.*, of the equation of the surface, is indeterminate, and, therefore, the polar of a point, by the same article, turns round a straight line; hence the locus of the pole is (*art.* 4) a straight line.

19. *The poles of a plane in reference to all surfaces of the second degree which touch seven fixed planes, lie in one and the same plane.*

The polar plane, by *art.* 17, turns round a point, and, therefore, the locus of the pole is (*art.* 3) a plane.

20. *Those diametral planes which are conjugate to parallel diameters, in reference to all surfaces of the second degree which pass through eight fixed points, intersect in one and the same straight line: also, the centres of such surfaces as touch eight fixed planes, lie in one and the same straight line.*

These are readily deduced from *articles* 16 and 18, when we reflect, that diametral planes of a surface of the second degree may be considered the polars of an infinitely distant point, and the centre of such surface, the pole of an infinitely distant plane.

Cor. If the surfaces pass through seven points, the diametral planes will intersect in the same point: also, if the surfaces touch seven planes, the centres will lie in the same plane.

21. *A curve surface of the second degree and a point are given, to find the locus of the points of intersection of the given surface with tangent planes drawn through the given point.*

Let $\gamma\beta\alpha$ be the point, and

$$az^2+by^2+cx^2+2ex=0,$$

the surface; then $z_1y_1x_1$ being a point in this surface, the equation of a tangent plane at this point is (*art.* 10)

$$az_1z+by_1y+(cx_1+e)\,x+ex_1=0.$$

This plane must be satisfied by the co-ordinates α, β, γ, of the given point, since this is a point in it. Hence we have

$$az_1\gamma+by_1\beta+(cx_1+e)\alpha+ex_1=0,$$

or, arranging for $z_1y_1x_1$, *etc.*,

$$a\gamma z+b\beta y+(c\alpha+e)x+e\alpha=0.$$

Eliminating z between this and the equation of the surface, there results, for the required locus, the following equation of a line of the second degree (writing m for $c\alpha+e$):

$$b(b\beta^2+a\gamma^2)y^2+(m^2+ac\gamma^2)x^2+2b\beta mxy$$
$$+2b\beta e\alpha y+2e(\alpha m+a\gamma^2)x+e^2\alpha^2=0.$$

From this and the preceding articles, we deduce the following corollaries:

Cor. 1. *A point without a given surface of the second degree and its polar in respect of the surface, are the vertex and directing plane respectively, of a conical surface of the second degree which envelopes the surface.*

Cor. 2. *If the centre (vertex) of a cone which envelopes a given surface, move upon a plane, the plane of the touching curve will turn round a point, but if the same vertex move upon a line, the plane of the touching curve will turn round a line: and, conversely, if the plane of the touching curve turn round a point, the centre of the enveloping cone will describe a plane, but if the plane of the touching curve turn round a line, the centre will describe a straight line.*

Cor. 3. *If the plane of contact move parallel to itself, the locus of the vertex of the enveloping cone will be a diameter of the surface.**

Cor. 4. *The vertices (centres) of the enveloping cones in reference to all surfaces of the second degree which touch eight fixed planes, lie in one and the same straight line.*

Cor. 5. *The vertices (centres) of the enveloping cones in reference to all surfaces of the second degree which touch seven fixed planes, lie in one and the same plane.*

22. *If the opposite faces of a hexahedron inscribed in a surface of the second order be produced to meet, the three lines of intersection will be in the same plane: Also, if an octahedron circumscribe the same surface (the angular points of the former being the points of contact of the latter), the three lines which join its opposite angular points, two and two, will pass through the same point.*

Let the equation of the surface be

$$az^2+by^2+cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx+1=0,$$

of which, one (suppose a) of the quantities a, b, *etc.*, is indeterminate, inasmuch as there are *eight* given points only in the surface.

Now if $z_1\, y_1\, x_1$ and $z_2\, y_2\, x_2$ be two opposite angular points of the circumscribed octahedron, then two intersecting faces of the inscribed hexahedron are (*art.* 11),

$$\begin{aligned}(az_1+fy_1+ex_1+g)z+(by_1+fz_1+dx_1+h)y&\\ +(cx_1+ez_1+dy_1+k)x+gz_1+hy_1+kx_1+1&=0,\\ (az_2+fy_2+ex_2+g)z+(by_2+fz_2+dx_2+h)y&\\ +(cx_2+ez_2+dy_2+k)x+gz_2+hy_2+kx_2+1&=0.\end{aligned}$$

Combining these by addition, and remembering that a is indeterminate, we get the following equations of a line in the plane of xy:

$$z=0,$$

$$\begin{aligned}&\{f(z_1+z_2)+b(y_1+y_2)+d(x_1+x_2)+2h\}y\\ &+\{e(z_1+z_2)+d(y_1+y_2)+c(x_1+x_2)+2k\}x\\ &+g(z_1+z_2)+h(y_1+y_2)+k(x_1+x_2)+2=0.\end{aligned}$$

In the same way it may be shewn that the other opposite faces intersect in the plane of xy. Wherefore, when a is the indeterminate, the three lines of intersection are in the same plane.

It may be proved, also, in a similar way, that when *any* one of the other quantities b, c, *etc.*, is the indeterminate, the three lines of intersection are in the same plane. The first theorem is consequently established.

Again, by *cor. art.* 12, the lines which join the opposite angular points of the circumscribed figure are the reciprocals of the lines of intersection of the opposite faces of the inscribed one; and (*art.* 5), "if three or more

* A particular case of this theorem, in reference to the ellipsoid, is given in the Diary for 1841, page 43.

straight lines lie in a plane, their reciprocal straight lines pass through the same point." This proves the second theorem.

Scholium.—These remarkable properties in reference to surfaces of the second order, correspond to those of *Pascal* and *Brianchon* in reference to lines of the second order. They are *new* so far as I know; for though several mathematicians have supposed that properties in space existed analogous to the celebrated theorems of Pascal and Brianchon in plano, no one, I believe, has yet given them. Chasles seems to have a notion that they are connected with the *tetrahedron*.

23. *If the three planes,*

$$z = m_1y + n_1x + p_1 \quad \ldots\ldots (27)$$
$$z = m_2y + n_2x + p_2 \quad \ldots\ldots (28)$$
$$z = m_3y + n_3x + p_3 \quad \ldots\ldots (29)$$

be mutually conjugate to one another (the perpendicular case included), there exists the relation

$$\frac{1}{m_1n_2} + \frac{1}{m_2n_3} + \frac{1}{m_3n_1} = \frac{1}{m_1n_3} + \frac{1}{m_2n_1} + \frac{1}{m_3n_2} \quad \ldots (30)$$

Let us refer the planes to central surfaces in general of the second degree, represented by the equation

$$az^2 + by^2 + cx^2 = 1 \quad \ldots\ldots (31)$$

Comparing this equation with (1) *article* 1, we get

$$a=a,\ b=b,\ c=c,\ d=0,\ e=0,\ f=0,\ g=0,\ h=0,\ k=0,\ l=-1;$$

the diameters, therefore, conjugate to the planes (27, 28, 29) are (*art.* 8)

$$by + m_1az = 0, \quad cx + n_1az = 0 \quad \ldots\ldots (32)$$
$$by + m_2az = 0, \quad cx + n_2az = 0 \quad \ldots\ldots (33)$$
$$by + m_3az = 0, \quad cx + n_3az = 0 \quad \ldots\ldots (34)$$

Now since the planes are conjugate, the line (32) is parallel to the plane (28); (33) is parallel to (29); and (34) to (27). From (28, 32) we readily get

$$z = \frac{bcp_2}{a(n_1n_2b + m_1m_2c) + bc};$$

in which, since (28, 32) are parallel, z must be *infinite*:

$$\therefore \quad a(n_1n_2b + m_1m_2c) + bc = 0 \quad \ldots\ldots (35)$$

Similarly, $\quad a(n_1n_3b + m_1m_3c) + bc = 0. \quad \ldots\ldots (36)$

And, $\quad a(n_2n_3b + m_2m_3c) + bc = 0 \quad \ldots\ldots (37)$

From (35, 36), $\quad b = \dfrac{cm_1(m_3 - m_2)}{n_1(n_2 - n_3)}$

Also, from (35, 37) $\quad b = \dfrac{cm_2(m_3 - m_1)}{n_2(n_1 - n_3)}$

$$\therefore \quad \frac{m_1(m_3 - m_2)}{n_1(n_2 - n_3)} = \frac{m_2(m_3 - m_1)}{n_2(n_1 - n_3)};$$

which, after reduction and transposition, gives the specified relation, viz.:

$$\frac{1}{m_1n_2} + \frac{1}{m_2n_3} + \frac{1}{m_3n_1} = \frac{1}{m_1n_3} + \frac{1}{m_2n_1} + \frac{1}{m_3n_2}.$$

Cor. *If $z_1\, y_1\, x_1$, $z_2\, y_2\, x_2$, $z_3\, y_3\, x_3$, be the points of contact of three conjugate tangent planes to a central surface of the second order; then*

$$\frac{1}{z_1y_3x_2}+\frac{1}{z_2y_1x_3}+\frac{1}{z_3y_2x_1}=\frac{1}{z_1y_2x_3}+\frac{1}{z_2y_3x_1}+\frac{1}{z_3y_1x_2}\ldots\ldots(38)$$

For these tangent planes (*art.* 10) are

$$az_1z+by_1y+cx_1x=1,$$
$$az_2z+by_2y+cx_2x=1,$$
$$az_3z+by_3y+cx_3x=1,$$

$\therefore\quad m_1=-\dfrac{by_1}{az_1},\quad n_1=-\dfrac{cx_1}{az_1}$, *etc.*, which being substituted in (30), give the relation in question.

24. *If the three lines in space,*

$$\left.\begin{aligned}y&=a_1z+a'\\x&=\beta_1z+\beta'\end{aligned}\right\}\ldots(39)\qquad\left.\begin{aligned}y&=a_2z+a''\\x&=\beta_2z+\beta''\end{aligned}\right\}\ldots(40)\qquad\left.\begin{aligned}y&=a_3z+a'''\\x&=\beta_3z+\beta'''\end{aligned}\right\}\ldots(41)$$

be mutually conjugate to one another (the case of perpendicularity included), there exists the relation

$$\frac{1}{a_1\beta_3}+\frac{1}{a_2\beta_1}+\frac{1}{a_3\beta_2}=\frac{1}{a_1\beta_2}+\frac{1}{a_2\beta_3}+\frac{1}{a_3\beta_1}\ldots\ldots\ldots\ldots(42)$$

Let the system be again referred to the general equation

$$az^2+by^2+cx^2=1.$$

Now it will be obvious that three diameters which are conjugate to three conjugate planes are mutually conjugate lines: hence, denoting the planes and their conjugate diameters as in last article, we have from (32, 33, 34) and (39, 40, 41),

$$a_1=-\frac{m_1a}{b},\quad a_2=-\frac{m_2a}{b},\ etc.,\ldots,\ \beta_1=-\frac{n_1a}{c},\quad \beta_2=-\frac{n_2a}{c},\ etc.\ldots$$

$$\therefore m_1=-\frac{a_1b}{a},\quad m_2=-\frac{a_2b}{a},\ etc.\ldots,\ n_1=-\frac{\beta_1c}{a},\quad n_2=-\frac{\beta_2c}{a},\ etc.\ldots$$

Substituting these values of $m_1, m_2\ldots,\ n_1, n_2\ldots$, in (35, 36, 37), we get the conjugate lines

$$a+ba_1a_2+c\beta_1\beta_2=0\ \ldots\ldots\ldots\ldots\ldots\ldots(43)$$
$$a+ba_1a_3+c\beta_1\beta_3=0\ \ldots\ldots\ldots\ldots\ldots\ldots(44)$$
$$a+ba_2a_3+c\beta_2\beta_3=0\ \ldots\ldots\ldots\ldots\ldots\ldots(45)$$

Operating with these as with (35, 36, 37) of last article, there results the enunciated relation (42). Hence since any three lines parallel to these will obviously have the same property, the theorem is fully established.

Cor. 1. *If two of three conjugate lines are given in position, the third is necessarily given in position.*

Cor. 2. *Let*

$$000,\ z_1y_1x_1,\ z_2y_2x_2,\ z_3y_3x_3,$$

be the co-ordinates of four points such that the three lines which join the first and each of the others are mutually conjugate to one another; then

$$\frac{1}{z_1y_3x_2}+\frac{1}{z_2y_1x_3}+\frac{1}{z_3y_2x_1}=\frac{1}{z_1y_2x_3}+\frac{1}{z_2y_3x_1}+\frac{1}{z_3y_1x_2}\ldots(46)$$

This will be obvious from the *Corollary* of the preceding article.

Cor. 3. *Three conjugate diameters to a surface of the second degree, may be expressed thus:*

$$az_1z_2+by_1y_2+cx_1x_2=0,$$
$$az_1z_3+by_1y_3+cx_1x_3=0,$$
$$az_2z_3+by_2y_3+cx_2x_3=0.$$

These are easily deduced from (43, 44, 45); z_1, z_2 *etc.*, in this as in *cor.* 2, being the co-ordinates of the extremities of three conjugate diameters.

Scholium.—The beautiful theorems contained in these articles (23, 24), are, I believe, *new*, the equations of condition in another form for *perpendicular lines or planes* only having been given. Those for perpendicular lines are

$$a_1a_2+\beta_1\beta_2+1=0,$$
$$a_1a_3+\beta_1\beta_3+1=0,$$
$$a_2a_3+\beta_2\beta_3+1=0,$$

which may be reduced to the general criterion given above, thus:

Taking the second and third, respectively, from the first, we get

$$a_1(a_2-a_3)+\beta_1(\beta_2-\beta_3)=0,\text{ or } a_1(a_2-a_3)=-\beta_1(\beta_2-\beta_3),$$

and, $a_2(a_1-a_3)+\beta_2(\beta_1-\beta_3)=0$, or $\beta_2(\beta_1-\beta_3)=-a_2(a_1-a_3)$,

$$\therefore\quad a_1\beta_2(a_2-a_3)(\beta_1-\beta_3)=a_2\beta_1(a_1-a_3)(\beta_2-\beta_3),$$

or, multiplying out and dividing by $a_1a_2a_3\,\beta_1\beta_2\beta_3$,

$$\frac{1}{a_1\beta_3}+\frac{1}{a_2\beta_1}+\frac{1}{a_3\beta_2}=\frac{1}{a_1\beta_2}+\frac{1}{a_2\beta_3}+\frac{1}{a_3\beta_1},$$

which is the criterion in general of *any* three conjugate lines, represented as in article 24. In a similar way may the equations of condition of perpendicular planes be reduced to the general criterion.

25. *The six co-ordinate axes of two systems of conjugate co-ordinates in space (the origin being the same) lie in a conical surface of the second degree.*

The equation of a conical surface, the vertex being the origin, is

$$az^2+by^2+cx^2+2dyx+2ezx+2fzy=0\ldots.(47)$$

Now, that the co-ordinate axes may lie in this surface, the equations $(y=0,\ x=0)$; $(z=0,\ x=0)$; $(z=0,\ y=0)$; must be severally satisfied, or $az^2=0$, $by^2=0$, $cx^2=0$, or $a=0, b=0, c=0$; so that the equation (47) becomes

$$\frac{d}{z}+\frac{e}{y}+\frac{f}{x}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(48)$$

The following conjugate lines will also lie in this surface, viz.:

$$\left.\begin{matrix}y=a_1z\\x=\beta_1z\end{matrix}\right\}\ldots(\gamma_1)\qquad\left.\begin{matrix}y=a_2z\\x=\beta_2z\end{matrix}\right\}\ldots(\gamma_2)\qquad\left.\begin{matrix}y=a_3z\\x=\beta_2z\end{matrix}\right\}\ldots(\gamma_3)$$

For, substituting from these equations in (48), we have

$$d+\frac{e}{a_1}+\frac{f}{\beta_1}=0,$$
$$d+\frac{e}{a_2}+\frac{f}{\beta_2}=0,$$
$$d+\frac{e}{a_3}+\frac{f}{\beta_3}=0.$$

Now if on eliminating d, e, f, from these, we get the criterion (art. 24) of

conjugate lines, it will be obvious that $(\gamma_1, \gamma_2, \gamma_3)$ satisfy (48), and, co sequently these lines will lie in that surface.

Eliminating d and e, there results the relation

$$\frac{f\left(\frac{1}{\beta_2}-\frac{1}{\beta_1}\right)}{\frac{1}{a_1}-\frac{1}{a_2}}=\frac{f\left(\frac{1}{\beta_3}-\frac{1}{\beta_1}\right)}{\frac{1}{a_1}-\frac{1}{a_3}},$$

or, reducing and multiplying out,

$$\frac{1}{a_1\beta_3}+\frac{1}{a_2\beta_1}+\frac{1}{a_3\beta_2}=\frac{1}{a_1\beta_2}+\frac{1}{a_2\beta_3}+\frac{1}{a_3\beta_1},$$

which is the criterion alluded to, and hence the surface (48) will contain th conjugate lines $(\gamma_1, \gamma_2, \gamma_3)$, which proves the theorem enunciated.

26. *The six co-ordinate planes of two systems of conjugate co ordinates in space (the origin being the same) touch a conical surfac of the second degree.*

If in the general equation (47) of *last art.*, of a conical surface, we p severally $z=0$, $y=0$, $x=0$, we obtain the equations

$$\left.\begin{aligned} by^2+cx^2+2dyx=0\\ az^2+cx^2+2ezx=0\\ az^2+by^2+2fzy=0\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (4$$

Each of these equations expresses two straight lines, the intersections the conical surface (47) with the co-ordinate axes. If, now, we put

$$d^2=\ bc,\qquad e^2=\ ac,\qquad f^2=\ ab,$$

which give

$$a=\pm\frac{ef}{d},\ b=\pm\frac{df}{e},\ c=\pm\frac{de}{f}\ldots\ldots\ldots\ldots\ldots(5$$

the expressions (49) then become squares, and, consequently the sec lines become tangents. The equation (47), by virtue of the *negative* val of a, b, c in (50), now becomes

$$\frac{z^2}{d^2}+\frac{y^2}{e^2}+\frac{x^2}{f^2}-\frac{2yx}{ef}-\frac{2zx}{df}-\frac{2zy}{de}=0\ldots\ldots($$

This surface will also touch the conjugate planes

$$\left.\begin{aligned} z=m_1y+n_1x\\ z=m_2y+n_2x\\ z=m_3y+n_3x\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

For, eliminating z between the first of these and (51) we get, a reduction, for the intersection of (51) with this plane, the equation

$$f^2(m_1e-d)^2y^2+e^2(n_1f-d)^2x^2+2\,ef(m_1n_1ef-d^2-m_1de-n_1df)yx=$$

and in order that this may be a square, we must have

$$\pm\,(m_1e-d)(n_1f-d)=m_1n_1ef-d^2-m_1de-n_1df.$$

Taking the second value, we get, after reduction,

$$ef-\frac{df}{m_1}-\frac{de}{n_1}=0\,;$$

$$\text{Similarly,}\quad ef-\frac{df}{m_2}-\frac{de}{n_2}=0,$$

$$\text{and}\qquad ef-\frac{df}{m_3}-\frac{de}{n_3}=0.$$

Eliminating d, e, f, from these, there results the criterion of conjugate planes (*art.* 23), viz.,

$$\frac{1}{m_1n_2}+\frac{1}{m_2n_3}+\frac{1}{m_3n_1}=\frac{1}{m_1n_3}+\frac{1}{m_2n_1}+\frac{1}{m_3n_2}.$$

Hence it will be obvious from what we have stated in the preceding article, that (51) touches also the co-ordinate planes (m).*

From the last two articles we readily deduce the following general theorem:

The co-ordinate axes of any two systems of conjugate co-ordinates in space (the origin being the same) lie in a conical surface of the second degree: also, the co-ordinate planes touch a conical surface of the second degree.

27. *If the angular points of any number of tetrahedra lie in a surface of the second degree, so that three corresponding plane faces pass through the same three points which are in a straight line; then will all the fourth plane faces pass through the same point, which is in the given line.*

Let us take the given line as the axis of z, and to simplify our operations, we shall refer the system to the general equation of central surfaces,

$$az^2+by^2+cx^2=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (n)$$

Then the vertices of four intersecting tangent cones to this surface, whose bases are the faces of the tetrahedra, being

$$z_1y_1x_1;\ z_2y_2x_2;\ z_3y_3x_3;\ z_4y_4x_4;$$

the faces (polars) will be denoted by the equations,

$$\begin{aligned}az_1z+by_1y+cx_1x&=1,\\ az_2z+by_2y+cx_2x&=1,\\ az_3z+by_3y+cx_3x&=1,\\ az_4z+by_4y+cx_4x&=1.\end{aligned}$$

But as these pass through the points $\gamma_1 00$; $\gamma_2 00$; $\gamma_3 00$; $v00$ (v we suppose *variable*); we have

$$az_1\gamma_1=1 \quad \therefore \quad z_1=\frac{1}{a\gamma_1}.$$

Similarly, $z_2=\dfrac{1}{a\gamma_2}$; $z_3=\dfrac{1}{a\gamma_3}$; $z_4=\dfrac{1}{av}$.

Hence, by substitution, the equations of the plane faces become,

$$\begin{aligned}z+by_1\gamma_1y+cx_1\gamma_1x-\gamma_1&=0,\\ z+by_2\gamma_2y+cx_2\gamma_2x-\gamma_2&=0,\\ z+by_3\gamma_3y+cx_3\gamma_3x-\gamma_3&=0,\\ z+by_4\,vy+cx_4\,vx-v\ &=0,\end{aligned}$$

which, since the co-ordinates z_2, y_2, x_2, *etc.*, have *all* values, denote all the tetrahedra in question.

Now, as the intersections of these planes lie in the surface (n), we may represent that surface thus:

$$\begin{aligned}&(z+by_1\gamma_1y+cx_1\gamma_1x-\gamma_1)(z+by_2\gamma_2y+cx_2\gamma_2x-\gamma_2)\\ &\quad+\lambda(z+by_3\gamma_3y+cx_3\gamma_3x-\gamma_3)(z+by_4vy+cx_4vx-v)=0\ldots.(52)\end{aligned}$$

* The particular cases of these theorems, when the axes are *rectangular*, (included also in articles 25, 26), have been discussed by Mr. Weddle in No. 3 of the Mathematician.

From the *identity* of (n) and (52) we have (equating the absolute terms and the coefficients of z) the relations

$$\gamma_1+\gamma_2+\lambda\gamma_3+\lambda v=0,$$
$$a(\gamma_1\gamma_2+\lambda\gamma_3 v)+(1+\lambda)=0.$$

Eliminating the arbitrary λ from these, we get

$$v=\frac{\gamma_1+\gamma_2-\gamma_3-a\gamma_1\gamma_2\gamma_3}{1-a(\gamma_1\gamma_3+\gamma_2\gamma_3-\gamma_1\gamma_2)},$$

which, as γ_1, γ_2, γ_3, are *constant*, shews that v has *one value only*, and, consequently, all the fourth plane faces pass through the same point in the axis of z.

28. *If the angular points of any number of tetrahedra lie in a surface of the second degree, so that three corresponding faces are parallel to three fixed lines in the same plane; then will all the fourth plane faces be parallel to a given straight line in this plane.*

Let the plane of the three given lines be the plane of xz; then the equations of these lines being

$$z+a_1x+\beta_1=0,\ z+a_2x+\beta_2=0,\ z+a_3x+\beta_3=0,$$

the equations of the four plane faces (the other notation the same as in last article) will be

$$z+\frac{by_1}{az_1}y+a_1x=\frac{1}{az_1},$$
$$z+\frac{by_2}{az_2}y+a_2x=\frac{1}{az_2},$$
$$z+\frac{by_3}{az_3}y+a_3x=\frac{1}{az_3},$$
$$z+\frac{by_4}{az_4}y+vx=\frac{1}{az_4}.$$

Hence we may represent the surface which contains the tetrahedra by the equation

$$\left(z+\frac{by_1}{az_1}y+a_1x-\frac{1}{az_1}\right)\left(z+\frac{by_2}{az_2}y+a_2x-\frac{1}{az_2}\right)$$
$$+\lambda\left(z+\frac{by_3}{az_3}y+a_3x-\frac{1}{az_3}\right)\left(z+\frac{by_4}{az_4}y+vx-\frac{1}{az_4}\right)=0,$$

which is identical with (n) of last article. Equating the coefficients of x and zx, we get the relations

$$a_1+a_2+\lambda a_3+\lambda v=0,$$
$$a(a_1a_2+\lambda a_3v)-c(1+\lambda)=0.$$

Eliminating λ

$$v=\frac{c(a_1+a_2-a_3)+aa_1a_2a_3}{c+a(a_1a_3+a_2a_3-a_1a_2)}.$$

Wherefore since v has one value only, the fourth planes are evidently all parallel to a given line in the plane of xz, the equation to which is

$$z+vx+\beta_4=0.$$

Scholium. The investigations contained in the last two articles, it will be seen, have reference to *central* surfaces only; an analogous mode of

investigation, however, is applicable; and similar results are obtained, when the equation of reference is that of curve surfaces in general; viz.,

$$ax^2 + by^2 + cx^2 + 2dxy + 2exz + 2fyz + 2gz + 2hy + 2kx + 1 = 0.$$

The value of v (*art.* 27) would in such case be

$$= \frac{\gamma_1 + \gamma_2 - \gamma_3 + 2g\gamma_1\gamma_2 + a\gamma_1\gamma_2\gamma_3}{1 + 2g\gamma_3 + a(\gamma_1\gamma_3 + \gamma_2\gamma_3 - \gamma_1\gamma_2)}.$$

The analogous theorems in plano, which are given below, may be interesting to the student.

29. *A conic section and a right line are given, and any number of quadrilaterals are inscribed in the conic section, to shew that if three corresponding sides of the quadrilaterals cut the given line in three given points, the fourth sides will also cut it in a given point: or if three corresponding sides be parallel to three given lines, the fourth sides will be parallel to a given line.**

Let us take the given line as the axis of x, and denote the conic section by the equation

$$ay^2 + bx^2 + 2cxy + 2dy + 2ex + 1 = 0 \ldots\ldots\ldots\ldots\ldots (53)$$

Then the poles of the sides of a quadrilateral inscribed in this conic section being

$$y_1x_1 ;\ y_2x_2 ;\ y_3x_3 ;\ y_4x_4 ;$$

the sides will be denoted by the equations (No. 3, page 128)

$$\begin{aligned}
(ay_1 + cx_1 + d)y + (bx_1 + cy_1 + e)x + dy_1 + ex_1 + 1 &= 0,\\
(ay_2 + cx_2 + d)y + (bx_2 + cy_2 + e)x + dy_2 + ex_2 + 1 &= 0,\\
(ay_3 + cx_3 + d)y + (bx_3 + cy_3 + e)x + dy_3 + ex_3 + 1 &= 0,\\
(ay_4 + cx_4 + d)y + (bx_4 + cy_4 + e)x + dy_4 + ex_4 + 1 &= 0.
\end{aligned}$$

But because these lines pass through the points a_1o; a_2o; a_3o; $t\,o$ (t we suppose *variable*); we have

$$\begin{aligned}
(bx_1 + cy_1 + e)a_1 + dy_1 + ex_1 + 1 &= 0,\\
(bx_2 + cy_2 + e)a_2 + dy_2 + ex_2 + 1 &= 0,\\
(bx_3 + cy_3 + e)a_3 + dy_3 + ex_3 + 1 &= 0,\\
(bx_4 + cy_4 + e)t + dy_4 + ex_4 + 1 &= 0.
\end{aligned}$$

Wherefore the equations of the sides become

$$\begin{aligned}
u_1 &= (ay_1+cx_1+d)y+(bx_1+cy_1+e)(x-a_1) = 0,\\
u_2 &= (ay_2+cx_2+d)y+(bx_2+cy_2+e)(x-a_2) = 0,\\
u_3 &= (ay_3+cx_3+d)y+(bx_3+cy_3+e)(x-a_3) = 0,\\
u_4 &= (ay_4+cx_4+d)y+(bx_4+cy_4+e)(x-t\ ) = 0,
\end{aligned}$$

which, as x_1, y_1, x_2, *etc.*, admit of *all* possible values, represent an indefinite number of such quadilaterals.

Hence the following will also denote the conic section (53):

$$u'_1u'_2 + \lambda u'_3u'_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

where $u'_1 = u_1$ divided by (bx_1+cy_1+e), $u'_2 = u_2$ divided by (bx_2+cy_2+e), and so on.

* Elegant solutions of these theorems in plano for *one* quadrilateral, are given in O'Brien's work on Analytical Geometry of two dimensions, recently published.

Equating, therefore, the absolute terms, and also the coefficients of x, (the *identical* equations (53, 54), we have the relations

$$b(a_1+a_2+\lambda a_3+\lambda t)+2\,e(1+\lambda)=0,$$
$$b(a_1a_2+\lambda a_3 t)-(1+\lambda)=0,$$

from which, eliminating λ, we get, finally,

$$t=\frac{a_1+a_2-a_3+2ea_1a_2+ba_1a_2a_3}{1+2ea_3+b(a_1a_3+a_2a_3-a_1a_2)}.$$

It follows, then, since t has one value only, that all the fourth sides pa through the same point in the given line.

The method of proving the second part of the theorem is exactly simil to that for the analogous one in space, given in *article* 28.

PROPOSITIONS ON THE CONIC SECTIONS.

[*Mr. James Dalmahoy, Edinburgh.*]

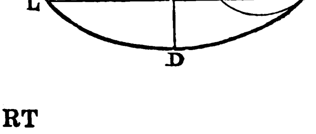
(Continued from page 125.)

PROP. III. THEOREM.

If at the vertex of the transverse axis of an ellipse or hyperbola, c at the vertex of a parabola, the circle of curvature be described, an also a rectilinear tangent; then, every chord of the conic section whic touches the circle and meets the tangent, will be divided harmonicall at the points of contact and intersection.

1. *The ellipse and hyperbola.* Let AB, CD, be the transverse ar conjugate axes; E the centre; F, G, the foci; H the centre of curvatu at the point A; and AX the tangent at the same point: also, let YL l any chord of the conic section, touching the circle of curvature in the poi M, and meeting the tangent in N: then, it is to be proved that

YM : ML : : NY : NL.



Draw the perpendiculars YP, YR, LQ, LS from Y and L upon AX and AB, and let K be the intersection of YR with the circle of curvature. Then it is well known that the centre of curvature of the point A is in AB; and that the centre itself lies wholly within the curve*; and its diameter AT is equal to the parameter of the axis AB. Also since YM is a tangent to the circle† we have

$$\text{YM}^2=\text{YR}^2-\text{KR}^2=\text{AR.RB}.\frac{\text{AT}}{\text{AB}}-\text{AR.RT}$$
$$=\text{AR.TB}.\frac{\text{AT}}{\text{AB}}\pm\text{AR.RT}.\frac{\text{AT}}{\text{AB}}\mp\text{AR.RT}.\frac{\text{AT}}{\text{AB}}-\text{AR.RT}.\frac{\text{TB}}{\text{AB}}$$
$$=\text{AR}^2.\frac{\text{TB}}{\text{AB}}.$$

* We have omitted the figure for the hyperbola, but the reader will easily constr it.—Eds.

† When the double sign is used, the upper one (as is usual) refers to the ellipse, a *the lower to the* hyperbola.

But since AB : AT : : BE^2 : DE^2, we have AB : TB : : BE^2 : FE^2, or

$$\frac{TB}{AB}=\frac{FE^2}{BE^2}, \text{ and } YM = AR.\frac{FE}{BE}=YP.\frac{FE}{BE}.$$

In a similar way it may be shewn that

$$LM = AS.\frac{FE}{BE}=LQ.\frac{FE}{BE}.$$

Hence, $$YM : ML : : YP.\frac{FE}{BE} : LQ.\frac{FE}{BE} : : YP : LQ;$$

and by the similar triangles NYP, NLQ, we have

$$YP : LQ : : NY : NL, \text{ and hence}$$
$$YM : ML : : NY : NL.$$

That is, the chord YL is divided harmonically in the points M and N.

A similar mode of investigation is applicable when the circle of curvature is described at the vertex of the *minor* axis.

2. *The parabola.* Consider YL as the chord of a parabola; then since YM is a tangent to the circle,

$$YM^2 = YR^2 - KR^2 = AR.AT - AR.RT = AR^2 = YP^2.$$

Hence $YM = YP$. Also,

$$LM^2 = LS^2 + AS.ST = AS.AT + AS.ST = AS^2 = LQ^2.$$

And hence $LM = LQ$; but

$$YP : LQ : : NY : NL, \text{ and hence } YM : LM : : NY : NL.$$

That is, the chord YL is divided harmonically in M and N.

Cor. In the ellipse and hyperbola, the distance of the vertex A from the directrix on the same side is known to be $GA.\frac{BE}{FE}$; wherefore the distance of the point Y from the same line is

$$GA.\frac{BE}{FE}+YM.\frac{BE}{FE}=GY.\frac{BE}{FE}; \text{ and hence } GA+YM=GY:$$

and in the same manner we find $GA+LM=GL$.

Also, by an investigation nearly similar, we obtain

$$GA \mp LM = FL, \text{ and } FA \mp YM = FY.$$

Wherefore, if a circle be described about the focus G with the radius GA, and another about the focus F with the radius FA, they will be touched (one internally and the other externally in the ellipse, and both externally in the hyperbola) by the circles which are described about the points Y and L with the respective radii YM and YL.

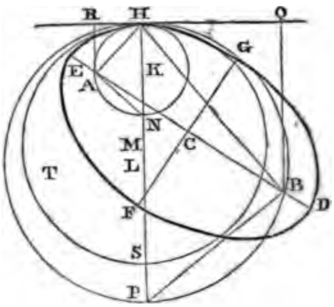
In the parabola it is evident that the tangent AX and the circle described about the focus with the radius AG, are touched by the circles described about Y and L with the respective radii YM and YL.

Schol. I would remark that the foregoing theorem was suggested to me by Leslie's demonstration of the ancient theorem in the Arbelon, with which he closes the second book of his *Geometrical Analysis*.

PROP. IV. THEOREM.

1. *The radius of curvature at a given point of an ellipse or hyperbola is harmonically divided by the circumferences of two circles, which touch the curve at the given point, and pass one through each focus.*

2. ***The radius of curvature at a given point of a parabola is bisected by the circumference of the circle which touches the curve in the given point and passes through the focus.***



1. *The ellipse and hyperbola.* Let EDFG be the curve, of which A and B are the foci, C the centre, ED and FG the transverse and conjugate axes; let H be the given point in the curve, M the centre of curvature at H; K and L the centres of the circles of contact at H which pass through the foci A and B: then the radius of curvature HM is divided harmonically at N and P by the circles HAN and HBP.

Draw the tangent QR at H, and from A and B the perpendiculars AR and BQ to it; also join BH, AH, BP, AN.

Now it is evident that the triangles HQB, PBH are similar, and that HRA, NAH are also similar. Whence

$$\frac{BH}{BQ}.\,BH = HP = \text{diameter of the circle HBP, and}$$

$$\frac{AH}{AR}.\,AH = HN = \text{diameter of the circle HAN.}$$

Also, $$\frac{BH^2}{BQ^2} = \frac{AH^2}{AR^2} = \frac{BH.AH}{BQ.AR}\,;\ \text{and } BH+AH = 2DC;$$

Whence $2DC\left(\dfrac{BH.AH}{BQ.AR}\right)^{\frac{1}{2}} = PH + HN$, the sum of the diameters of the circles drawn through the foci.

The product of the same diameters is, obviously,

$$PH.HN = \frac{BH^2.AH^2}{BQ.AR}\,;$$

Hence

$$\frac{2PH.HN}{HP+HN} = \frac{(BH.AH)^{\frac{3}{2}}}{DC(BQ.AR)^{\frac{1}{2}}} = \frac{(BH.AH)^{\frac{3}{2}}}{FC.CD} = MH,$$

the radius of curvature at H; and from this it follows that

$$HN : HP :: MH - HN : PH - HM,$$

which establishes the proposition for the ellipse.

The only difference in the detail of the proof for the hyperbola is that we have in this case

$$2DC\left(\frac{BH.HA}{BQ.AR}\right)^{\frac{1}{2}} = HP - HN,$$

and the result is

$$\frac{2HP.HN}{HP-HN} = MH, \text{ and } HN : HP :: HM - HN : HP + HM.$$

2. *The parabola.* In this case, obviously, we have $\dfrac{AH^2}{AR} = HN$, and $\dfrac{2AH^2}{AR} = HM$, the radius of curvature at H: whence $2HN = HM$, or the radius of curvature is bisected by the circle HAN.

We might have anticipated this as the limiting case of the ellipse and hyperbola, and the known limiting case of the harmonical division of a line.

Cor. In the ellipse and hyperbola, the equations may be expressed thus:

$$\frac{1}{\mathrm{HN}} \pm \frac{1}{\mathrm{HP}} = \frac{2}{\mathrm{HM}}.$$

PROP V. THEOREM.

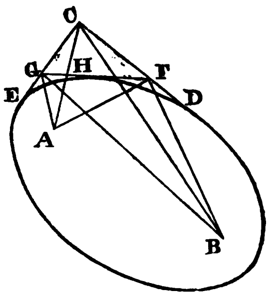
If the sides of a triangle touch a conic section,

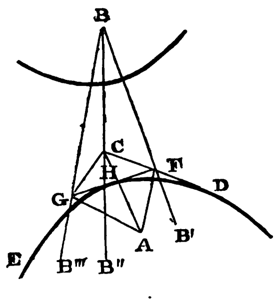
1. *In the parabola, the angle subtended at the focus by any side of the triangle, is equal to the supplement of the angle* contained by the other two sides.*

2. *In the ellipse, the sum, and in the hyperbola the difference of the angles, subtended at the foci, by any side of the triangle, is equal to the supplement of the angle contained by the other two sides.*

1. *The parabola.* This case of the theorem is already well known†, and need not be demonstrated.

2. *The ellipse and hyperbola.* Let the sides of the triangle FCG touch at the points D, H, E, the conic section of which the foci are A and B: draw lines from A and B to the angular points F, C, G, and, in the hyperbola, produce the lines BF, BC, BG indefinitely towards B′, B″, B‴: it is





to be proved that FAG ± FBG‡ $= \pi -$ DCE: FAC ± FBC $= \pi -$ FGE: GAC ± GBC $= \pi -$ DFG.

By a property common to both curves,

$$\mathrm{GFA} = \mathrm{DF}\frac{\mathrm{B}}{\mathrm{B}'}\ddagger = \mathrm{FC}\frac{\mathrm{B}}{\mathrm{B}''} \pm \mathrm{FBC}.$$

Also, $$\mathrm{FGA} = \mathrm{EG}\frac{\mathrm{B}}{\mathrm{B}'''} = \mathrm{GC}\frac{\mathrm{B}}{\mathrm{B}''} \pm \mathrm{GBC}.$$

Hence $$\mathrm{AFG} + \mathrm{AGF} = \pi - \mathrm{FAG} = \mathrm{DCE} \pm \mathrm{FBG};$$

and hence again, $$\pi - \mathrm{DCE} = \mathrm{FAG} \pm \mathrm{FBG}.$$

Again, $\mathrm{EGA} = \mathrm{GCA} + \mathrm{GAC}$, and $\mathrm{EG}.\frac{\mathrm{B}}{\mathrm{B}'''} = \mathrm{GC}\frac{\mathrm{B}}{\mathrm{B}''} \pm \mathrm{GBC}$,

but, $\mathrm{EG}\frac{\mathrm{B}}{\mathrm{B}'''} = \mathrm{FGA}$, and $\mathrm{GC}\frac{\mathrm{B}}{\mathrm{B}''} = \mathrm{FCA}$;

* The angle referred to is that the sides of which are touched internally by the conic section.

† Hutton's Mathematics, 12th edition, vol. ii., page 118, and Preface, p. 5.

‡ Where the double sign or double letter occurs, the upper one applies to the ellipse and *the lower to the hyperbola.*

hence $\quad \text{EGA} + \text{EG}\dfrac{\text{B}}{\text{B}'''} = \text{EGF} = \text{GCF} + \text{GAC} \pm \text{GBC} = \text{GCF} + \text{GFC};$

and therefore $\quad \text{GAC} \pm \text{GBC} = \text{GFC} = \pi - \text{DFG}.$

In a similar manner it may be proved that $\text{FAC} \pm \text{FBC} = \pi - \text{FGE}$.

It may further be remarked that if the ellipse touched the side FG internally instead of externally, the result would have been $\text{FAG} + \text{FBG} = \pi + \text{DCE}$.

Cor. 1. If tangents be applied at the extremities of any diameters of an ellipse or hyperbola, and a third tangent be drawn to cut these; in the ellipse, the sum, and in the hyperbola the difference of the angles subtended at the foci by the intercepted portion of the tangent, is equal to two right angles.

Cor. 2. If tangents be applied at the extremities of any diameter, and another tangent be drawn to cut these, in the ellipse, the sum of the angles subtended at the foci by the intercepted portion of the tangent, is equal to two right angles, and in the hyperbola, the angles subtended at the foci are equal to each other.

Cor. 3. From the property in Cor. 2, as respects the ellipse, may be deduced the following:—If a parallelogram be described about an ellipse, touching the extremities of two conjugate diameters, the rectangle under the sum of the distances of the foci from one of the angles and the sum of their distances from either of the adjacent angles, is equal to the rectangle under the semitransverse axis and the perimeter of the parallelogram.

Cor. 4. If two tangents to a conic section be produced to meet, the supplement of the angle contained by them is, in the parabola, equal to half the angle subtended at the focus by the chord of contact; and in the ellipse, it is half the sum, and in the hyperbola half the difference of the angles subtended at the foci, by the chord of contact.

Cor. 5. In the ellipse or hyperbola, twice the angle contained by the tangents at the extremity of a focal chord, plus or minus the angle subtended by that chord at the other focus, is equal to two right angles.

In the parabola the angle subtended at the distant focus being evanescent, the angle contained by the tangents at the extremities of the focal chord is, as already known, a right angle.

APPLICATION OF ALGEBRA TO THE MODERN GEOMETRY.*

[*Mr. Robert Finlay, Professor of Mathematics and Natural Philosophy in Manchester New College.*]

(*Continued from p.* 265.)

XV.

In all that precedes, I have only considered intersections and contacts which range upon straight lines. It is evident, however, that the lines of intersection and contact may be any given curves. Thus if $u = 0$ be a curve of the n^{th} degree, and if $v = 0$, $v' = 0$ be any two curves of the m^{th} degree,

* The reader is requested to make the following corrections in the former part of this paper.

Page 258—In equation (6) at the bottom of the page, for $\text{C}y^2$ read $\text{C}x^2$.

Page 259—line 11 from the top, for points read point.

line 14 from the bottom, for $\text{A}x^2$ read $\text{A}y^2$.

line 5 from the bottom, for $y'u'^{\frac{1}{2}}$ read $y''u'^{\frac{1}{2}}$.

line 2 from the bottom, for points read point.

where m is not greater than $\frac{n}{2}$; then $u = qvv'$ (where q is an arbitrary constant) will be a curve of the n^{th} degree, passing through the points in which the curves $v=0$, $v'=0$ intersect the curve $u=0$. Again, when the curves $v=0$, $v'=0$ coincide, the 2 mn points of intersection unite in mn points of contact; so that the equation $u = qv^2$ represents a curve of the n^{th} degree touching the curve $u=0$ at mn points on *the curve of contact* $v=0$.

XVI.

To find the locus of a point P, such that if a straight line be drawn from it to a given point A, meeting a given curve of the third degree in Q_1, Q_2, Q_3, and a given curve of the second degree in Q', Q'': then

$$\frac{PQ_1 \times PQ_2 \times PQ_3}{AQ_1 \times AQ_2 \times AQ_3} \text{ shall be to } \frac{PQ' \times PQ''}{AQ' \times AQ''} \text{ in a given ratio.}$$

Let the equations of the given curves referred to any rectangular axes passing through the given point be

$$Ay^3+Bxy^2+Cx^2y+Dx^3+Ey^2+Fxy+Gx^2+Hy+Kx+1=0 \quad (a)$$
$$A'y^2+B'xy+C'x^2+D'y+E'x+1=0 \quad (b)$$

and put $AP=r$, $xAP=\theta$, $AQ'=\rho'$, $AQ''=\rho''$, $AQ_1=\rho_1$, $AQ_2=\rho_2$, $AQ_3=\rho_3$: then by the condition of the question

$$\frac{(r-\rho_1)(r-\rho_2)(r-\rho_3)}{\rho_1\rho_2\rho_3} = m\,\frac{(r-\rho')(r-\rho'')}{\rho'\rho''}, \text{ or}$$

$$\frac{r^3}{\rho_1\rho_2\rho_3} - \left(\frac{1}{\rho_1\rho_2}+\frac{1}{\rho_2\rho_3}+\frac{1}{\rho_1\rho_3}\right)r^2 + \left(\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}\right)r - 1$$
$$= m\left\{\frac{r^2}{\rho'\rho''} - \left(\frac{1}{\rho'}+\frac{1}{\rho''}\right)r + 1\right\} \quad (c)$$

Now the polar equations of the given curves being

$$(A\sin^3\theta+B\sin^2\theta\cos\theta+C\sin\theta\cos^2\theta+D\cos^3\theta)\rho^3+(E\sin^2\theta+F\sin\theta\cos\theta+G\cos^2\theta)\rho^2$$
$$+(H\sin\theta+K\cos\theta)\rho+1=0,$$
$$(A'\sin^2\theta+B'\sin\theta\cos\theta+C'\cos^2\theta)\rho^2+(D'\sin\theta+E'\cos\theta)\rho+1=0;$$

since ρ', ρ'' are the roots of the latter equation, and ρ_1, ρ_2, ρ_3 those of the former, we obtain by the theory of equations,

$$\frac{1}{\rho'\rho''} = A'\sin^2\theta+B'\sin\theta\cos\theta+C'\sin^2\theta;$$
$$\frac{1}{\rho'}+\frac{1}{\rho''} = -(D'\sin\theta+E'\cos\theta);$$
$$\frac{1}{\rho_1\rho_2\rho_3} = -(A\sin^3\theta+B\sin^2\theta\cos\theta+C\sin\theta\cos^2\theta+D\cos^3\theta);$$
$$\frac{1}{\rho_1\rho_2}+\frac{1}{\rho_2\rho_3}+\frac{1}{\rho_1\rho_3} = E\sin^2\theta+F\sin\theta\cos\theta+G\cos^2\theta;$$
$$\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3} = -(H\sin\theta+K\cos\theta);$$

hence, if x, y be the co-ordinates of P, so that $x = r\cos\theta$, $y = r\sin\theta$, we obtain, by substituting these expressions in equation (c),

$$Ay^3+Bxy^2+Cx^2y+Dx^3+Ey^2+Fxy+Gx^2+Hy+Kx+1$$
$$=-m(A'y^2+B'xy+C'x^2+D'y+E'x+1)\,..(c'):$$

consequently the locus of P is a curve of the third degree passing through the six points, real or imaginary, in which the curves (a) and (b) intersect.

Generally, if a straight line AP, drawn through a given point A, meet a given curve of the n^{th} degree in $Q_1, Q_2, \ldots . Q_n$, and a curve of the m^{th} degree in R_1, R_2 ...... R_m, and if the point P be taken so that $\dfrac{PQ_1\times PQ_2 .. \times PQ_n}{AQ_1\times AQ_2 .. \times AQ_n}$ is to $\dfrac{PR_1\times PR_2 .. \times PR_m}{AR_1\times AR_2 .. \times AR_m}$ in a given ratio; then it is evident that the locus of P is a curve of the n^{th} degree (n being greater than m) passing through the mn points of intersection of the two given curves.

XVII.

If (C), (C′), (C″) be three conic sections which pass through the same four points (real or imaginary,) and if an arbitrary transversal be drawn from a fixed point A, meeting them in P and P′, Q and Q′, R and R′ respectively: then (XVI)

$$\frac{PQ\times PQ'}{AQ\times AQ'} : \frac{PR\times PR'}{AR\times AR'} :: m : 1,\quad \frac{P'Q\times P'Q'}{AQ\times AQ'} : \frac{P'R\times P'R'}{AR\times AR'} :: m : 1;$$

$$\therefore PQ\times PQ'\times P'R\times P'R' = P'Q\times P'Q'\times PR\times PR':$$

consequently the six points, P, P′, Q, Q′, R, R′ are in involution.

This extension of the well-known theorems of *Pappus* and *Desargues* is due to *Sturm*. It is evidently a simple corollary from the general theorem demonstrated in the last number.

XVIII.

Again, if (C), (C′), (C″) be three curves of the third degree, each of which passes through the nine points of intersection of the other two, and if an arbitrary transversal be drawn through any fixed point A, meeting the first in P, P′, P″, the second in Q, Q′, Q″, and the third in R, R′, R″; we shall evidently have (XVI), (m being a constant)

$$PQ\,.\,PQ'.\,PQ'' : PR\,.\,PR'\,.\,PR'' :: m.AQ.AQ'.AQ'' : AR.AR'.AR'',$$
$$P'Q.P'Q'.P'Q'' : P'R.\,P'R'.P'R'' :: m.AQ.AQ'.AQ'' : AR.AR'.AR'',$$
$$P''Q.P''Q'.P''Q'' : P''R.P''R'.P''R'' :: m.AQ.AQ'.AQ'' : AR.AR'.AR''.$$

From the first two of these analogies we deduce

$$PQ.PQ'.PQ''.P'R.P'R'.P'R''=P'Q.P'Q'.P'Q''.PR.PR'.PR'' \quad ...(a):$$

a relation which from analogy may be called *the involution of eight points.*

The above analogies show that any two of the points P, P′, P″, together with the six points Q, Q′, Q″, R, R′, R″ form a system of eight points in involution; and in like manner it may be shewn that any two of the points Q, Q′, Q″, together with the other six, form a system of eight points in involution; and that any two of the points R, R′, R″, together with the other six, form a similar system. Hence, equation (a) is one of a system of nine equations which hold among the segments of the transversal, and from which many others may be obtained by elimination.*

* By taking these nine equations in combinations of threes, we shall obtain four equations from each combination by multiplication, making in all twelve equations of the form

$$PQ'^2\times P'R^2\times QR'^2\times PQ''\times QR''\times RP''\times P'R''\times Q'P''\times R'Q''$$
$$=P'Q^2\times PR'^2\times Q'R^2\times P''Q\times Q''R\times R''P\times P''R'\times Q''P'\times R''Q'.$$

XIX.

Generally, if (C), (C′) (C″) be three curves of the n^{th} degree, each of which passes through the points of intersection of the other two, and if an arbitrary transversal be drawn through any fixed point A, meeting the first curve in P_1, $P_2 .. P_n$, the second in Q_1, Q_2, $Q_3 .. Q_n$, and the third in R_1, $R_2, .. R_n$, we shall have

$$P_1Q_1 \times P_1Q_2 .. \times P_1Q_n : P_1R_1 \times P_1R_2 .. P_1R_n :: m \times AQ_1 \times AQ_2 ... \times AQ_n : AR_1 \times AR_2 .. AR_n,$$

$$P_2Q_1 \times P_2Q_2 .. \times P_2Q_n : P_2R_1 \times P_2R_2 .. P_2R_n :: m \times AQ_1 \times AQ_2 .. \times AQ_n : AR_1 \times AR_2 .. AR_n;$$

$$\therefore P_1Q_1 \times P_1Q_2 .. P_1Q_n \times P_2R_1 \times P_2R_2 .. P_2R_n = P_2Q_1 \times P_2Q_2 .. P_2Q_n \times P_1R_1 \times P_1R_2 .. \times P_1R_n \ldots\ldots (a)$$

a property which may be called *the involution of* $2n+2$ *points*. In like manner it may be shown that any two of the points Q, together with the points P and R, form a system of $2n+2$ points in involution; and that any two of the points R, together with P and Q, form a similar system: hence (a) belongs to a system of $\frac{3n(n-1)}{2}$ equations, which express the relations of involution that exist among the segments intercepted by the $3n$ points in which the transversal meets the three curves.

XX.

Lastly, let (C′) be a curve of the n^{th} degree, (C″) one of the m^{th} degree, and (C) a curve of the n^{th} degree, passing through the intersections of (C′) and (C″): then if an arbitrary transversal be drawn from a fixed point A, meeting (C) in P_1, $P_2, \ldots . P_n$, (C′) in Q_1, Q_2, $Q_3 \ldots . Q_n$, and (C′) in R_1, $R_2 .. R_m$, we shall have

$$P_1Q_1 \times P_1Q_2 .. \times P_1Q_n : P_1R_1 \times P_1R_2 .. \times P_1R_m :: a \times AQ_1 \times AQ_2 .. \times AQ_n : AR_1 \times AR_2 .. \times AR_m;$$

$$P_2Q_1 \times P_2Q_2 .. \times P_2Q_n : P_2R_1 \times P_2R_2 .. \times P_2R_m :: a \times AQ_1 \times AQ_2 .. \times AQ_n : AR_1 \times AR_2 .. \times AR_m:$$

$$\therefore P_1Q_1 .. \times P_1Q_n \times P_2R_1 \times P_2R_2 .. \times P_2R_m = P_2Q_1 \times P_2Q_2 .. \times P_2Q_n \times P_1R_1 \times P_1R_2 .. \times P_1R_m \ldots\ldots (a')$$

This relation may be called *the involution of* $m+n+2$ *points*.

By similar processes we should find that the segments into which the transversal is cut by the three curves are connected by $n(n-1)$ relations of involution of $m+n+2$ points, and $\frac{m(m-1)}{2}$ relations of involution of $2n+2$ points.

ON THE TRANSFORMATION OF ALGEBRAIC EQUATIONS.

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1. On casting the eye over the operations at pages 83 and 84 of this publication, it will be seen that, it is in the dependence of $\pi'(v)$, $\pi''(v)$ *etc.*, on $\pi(v)$, and on one another, that their principal utility consists. Without this, the relation given by $\pi(v)=0$ would be a barren, though curious fact; but with it, we are furnished with transformations which are expressed with equal ease for any values of λ and λ', and which, by affording us

considerable latitude of choice in that respect, enable us to satisfy otl conditions. This dependence I before pointed out; it now remains demonstrate it. I shall, however, first give an outline of the principles which such transformation depends, and, in so doing, shall use π, ϕ, *et* instead of $\pi(v)$, $\phi(v)$, *etc.*

2. Let $F(p) = 0$......................................(

be a relation existing among the coefficients of an equation, and

$$f(v) = 0 \ldots\ldots ($$

a relation between its roots. Express (1) in symmetric functions of t roots and eliminate one of them, (v_1 for instance) between (1) and (2), a we have the resulting equation

$$f(v_2, v_3 \ldots v_n) = 0 \ldots\ldots ($$

then, when this last equation is identical, (1) and (2) denote interchangeal conditions. Now the above functions in (1) and (2) may, one or both, composed of any number of functions Ψ, σ, *etc.* any how connected. V shall here limit our investigations to one of their simplest forms, as that w suffice to elucidate the results which I have arrived at, at the place abc cited.—Let then, (1) be homogeneous and of the $(n-1)^{\text{th}}$ degree w respect to the roots; let (2) be linear and equivalent to

$$v_1 + av_2 + bv_3 + \ldots . lv_n = \phi_1 = 0 \ldots\ldots ($$

also let (3) be identical; then $F(p) = k_1\phi_1$. Again, (3) being identica zero, we may interchange v_2 and $v_3 \ldots . v_n$ and v_2 in (1) and (2), hence obtain $k_1 = k_2\phi_2$, and similarly we arrive at $k_{n-1} = k\phi_{n-1}$, and therefo $F(p) = k\pi$, where π is the product of some* $n-1$ of the quantities ϕ, *et* and k is free from v, and, consequently, π symmetric since (1) is so.

3. To apply these remarks to the reduction of equations of the first fo degrees respectively to the forms (1), (2), and (3) of p. 83, the above equati (1) must be assumed so as to fulfil the requisite conditions. We shall th obtain from (3) such values of a, b, *etc.*, as will render the functions ϕ, h made use of, identical with those of p. 83. The importance of this deco position of (1) will be clear from the next paragraph. From what is kno of the roots of equations of those degrees the reader will easily show himself that $\phi = 0$ is inconsistent with any but the forms of p. 83. It w be inferred from the last paragraph, that the method is not limited to the cases—in fact, the above simple form of it furnishes us, (as I shall show another opportunity,) with a test for determining the admissibility of certa assumptions respecting the forms of the roots of equations of the higl degrees.

4. The next point, which (as it will not improbably be useful in so future developement of the above theory,) I shall treat in all its generali is to show that "when π is symmetric, π', π'', *etc.*, are so likewise."

Now, (see pp. 83, 84) $\pi' = \Sigma(\phi_1\phi_2 \ldots \phi_{n-2}, \phi_{n-1})$, and consists of n- terms, which we may respectively denote by ω_1, $\omega_2 \ldots \omega_{n-1}$. On consideri one of them, ω_1 for instance, separately, we see that it is of the form

$$\ldots + m_1 x^{p-1} x_2^q \ldots x_m^r x'_1 + m'_1 x_1^p x_2^{q-1} \ldots x_m^r x'_2 + \ldots + m''_1 x_1^p x_2^q \ldots x_m^{r-1} x'_m -$$

and when $x'_1 = x_1$, $x'_2 = x_2$, *etc.*, the above terms become similar, and th

* There may, it is true, be different systems of values of ϕ in some cases, but, as in th which I shall here notice there is only one, and, consequently no discretion to be exercised *the choice of* values of ϕ, it is unnecessary to say more on the point.

sum equals $\Sigma(m_1)x_1^p\, x_2^q\, ..x_m^r$. In like manner we shall have the sum of other terms equal $\Sigma(\mu_1)x_1^q\, x_2^p\, ..\, x_m^r$, *etc.*; but, in this case $\omega_1 = \pi$, which is, by hypothesis, symmetric, hence

$$\Sigma(m_1) = \Sigma(\mu_1) = etc, \text{ so } \omega_2 \text{ gives}$$
$$\Sigma(m_2) = \Sigma(\mu_2) = etc. .. \ \omega_3, etc.....$$
$$\ldots\ldots\ldots\ldots \text{ and } \omega_{n-1} \ldots\ldots$$
$$\Sigma(m_{n-1}) = \Sigma(\mu_{n-1}) = etc.$$

and in each line, from the nature of the functions employed, the quantities are identical though they occur in a different order, and all the sums included under the symbol Σ are equal. Let r be the number of quantities in each group, then m being the number of groups in each line, we have rm quantities combined r together $(n-1)m$ ways and in all cases, so that the sum is the same and r never greater than $n-1$. Now one, at least, of the quantities is changed in each group; hence, when r is less than $n-1$, we ascertain, by the aid of rm of these groups, that the quantities are recurrent and have only r different values; and since in the remaining $(n-r-1)m$ groups one, at least, of these quantities must enter, at least, twice (the sum remaining the same in all cases) we have two of these last values equal; and since, when r is less than $n-1$, m is greater than r, we may, similarly, infer that all the values are equal, and so far as these terms are concerned, π symmetric.

5. Again, when $r=n-1$, we may infer, as before, that the quantities have only $n-1$ different values; and since, in each value of ω the same term in x, *etc.*, will have a different coefficient π', which equals the sum of the values of ω, will, so far as these terms are concerned, have the same coefficient, and, consequently, be symmetric. Hence, by combining this paragraph and the last, we see that, π being symmetric, π' is so likewise; and similarly, we might show that π'', *etc.*, also are symmetric.

6. We have now a rule for the derivation of π' from π, viz.:—"change one λ into λ', as at p. 84, and give the different terms multipliers such that when $\lambda' = \lambda$, π' may be equal to $(n-1)\pi$:" or, in other words, "multiply by $n-1$, and divide each term by the number of different roots which enter into that term." Similar principles apply to the derivation of π'' from π', *etc.*

7. The neglect of the latter branch of this rule has led me into an error at line 3 of p. 84, and at line 5 of paragraph 5 of p. 195. To correct the the former, we must multiply the first term of the coefficient of z by 2—for the latter, we must multiply the first term of the right hand side of the equation by 3, and the second and third by $\frac{3}{2}$ in conformity with the rule.

8. Although I have not expressly extended these remarks to the last case of p. 84, (see the last line of paragraph 5 of p. 195,) yet enough has been said to render such extension easy in that and other similar cases. There remains, however, one observation arising out of par. 8 of p. 196. Let us generalize the operation [1] there made use of, and denote by it the process of multiplying a quantity by p_1, and subtracting from the product n times a complementary quantity, n being the degree of a given equation. Then putting the complementary quantity within, and the other without, the bracket, the equations (1), (2), and (3) of p. 83 depend respectively upon the three following

$$p_1[0] = 0, \quad p_1[p_2] = 0, \quad p_1[p_2]'[2p_3] = 0,$$

where the accent is used to denote that the expression on the left of it is to be treated as one quantity, and where it will be observed that the coefficient of p_3 is a factor of n.*

9. In my next paper it will be seen that, by combining the method of par. 2 of p. 113 with that of Mr. Jerrard, we may extend the latter to equations of degrees lower than those fixed by Sir W. R. Hamilton as the limits of its original application; and that the analysis of which I have given the outlines at pp. 113-116, seems free from any such limitations whatsoever.

I shall conclude by observing that if, for the roots of the equation of the n^{th} degree, we assume a form similar to that which Mr. Murphy (Phil. Trans., 1837) has assumed for those of the equation of the 5^{th} degree, viz.

$$x = z_1 + az_2 + a^2z_3 + \ldots\ldots + a^{n-1}z_n\,,$$

where we must give a all its values successively, we find, by the properties of the roots of unity, that $\phi_1(x) = nz_n$, hence, if $\phi_1 = 0$, $z_n = 0$. We shall, in order to avail ourselves of this, reverse our processes and determine F from f.

(*a*) A slight delay having taken place in the publication of this paper, I take the opportunity of adding a few remarks to it. The letter h attached to a figure refers to Sir W. R. Hamilton's paper, in the 6th Rep. of the Brit. Association, and m to this volume of the *Mathematician*. In the latter case I have slightly modified the notation so as to make it conformable to Sir William's. The 2nd of the following equations is, in principle, the same as that which Sir W. denotes by (36),

(*b*) If $y = \Sigma(\Lambda' x^{\lambda'}) + \Sigma(\mathrm{M}' x^{\mu'}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)h$

we must exclude the cases in which

$$\Sigma(\mathrm{M}'x^{\mu'}) = a\Sigma(\Lambda'x^{\lambda'}) + \lambda\mathrm{X} \ldots\ldots\ldots\ldots\ldots (36)h$$

and, if n be a positive integer,

$$x^n = s_0^{(n)} + s_1^{(n)}x + s_2^{(n)}x^2 + \ldots + s^{(n)}_{m-1}x + \mathrm{L}^{(n)}\mathrm{X} \ldots\ldots (44)h$$

where $s_0^{(n)}, s_1^{(n)} \ldots s^{(n)}_{m-1}$ are symmetric functions of the roots of the proposed equation $\mathrm{X} = 0$.

(*c*) Hence, to effect the transformation at paragraph 8, p. 115 m,

$$\text{Let } \left.\begin{array}{l} \Lambda' s_0^{(\lambda')} + \Lambda'' s_0^{(\lambda'')} = p_0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Lambda' s_{m-1}^{(\lambda')} + \Lambda'' s_{m-1}^{(\lambda'')} = p_{m-1} \end{array}\right\} (46)h \quad \left.\begin{array}{l} \mathrm{M}' s_0^{(\mu')} + \mathrm{M}'' s_0^{(\mu'')} = p'_0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{M}' s_{m-1}^{(\mu')} + \mathrm{M}'' s_{m-1}^{(\mu'')} = p'_{m-1} \end{array}\right\} (47)h$$

$$\Lambda'\mathrm{L}^{(\lambda')} + \Lambda''\mathrm{L}^{(\lambda'')} = \Lambda \ldots (48)h,\ \ \mathrm{M}'\mathrm{L}^{(\mu')} + \mathrm{M}''\mathrm{L}^{(\mu'')} = \mathrm{M} \ldots (49)h,\ \ \Lambda + \mathrm{M} = \mathrm{L} \ldots (50)h$$

$$\text{Then } \Sigma(\Lambda'x^{\lambda'}) = p_0 + p_1x + p_2x^2 + \ldots + p_{m-1}x^{m-1} + \Lambda\mathrm{X} \ldots (51)h$$

$$\Sigma(\mathrm{M}'x^{\mu'}) = p'_0 + p'_1x + p'_2x^2 + \ldots + p'_{m-1}x^{m-1} + \mathrm{MX} \ldots (52)h$$

and (21)h becomes

* I shall, probably, enlarge upon this on another occasion.

$$y=p_0+p'_0+(p_1+p'_1)x+(p_2+p'_2)x^2+\ldots\ldots+(p_{m-1}+p'_{m-1})x^{m-1}+LX \quad \ldots\ldots(53)h$$

and to avoid (36)h we must avoid the proportionality of $p_0, p_1 . . p_{m-1}$ to $p'_0, p'_1 . . p'_{m-1}$.

(*d*). Let, therefore, $\frac{p'_{m-1}}{p_{m-1}} = p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(54)h$

$$p'_0 = pp_0 + q_0\,,\; p'_1 = pp_1 + q_1 \ldots . p'_{m-2} = pp_{m-2} + q_{m-2} \ldots . (56)h$$

then, we have only to take care, that the $m-1$ quantities $q_0, q_1 . . q_{m-2}$ shall not all vanish.

(*e*). Next, put the equation (16)*m*., page 115, under the form

$$B = B'_{2,0} + B'_{1,1} + B'_{0,2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(25)h$$

then, the effect of the analysis at paragraphs 3 and 8 of pp. 114-5, *m*. is to decompose (25)h into

$$B'_{1,1}+B'_{0,2}+Z'_{2,0}=0 . . (16.a), \text{ and } B'_{2,0}-Z'_{2,0}=0 . . (16.b),$$

adding these last, we obtain (25)h again. This latter we may and shall accordingly take for one of our *final* equations, (16.b) being the other. By a transformation, similar to that at page (304)h, we thence obtain the resulting equations,

$$(1+p)^2 B''_{2,0}+(1+p)B''_{1,1}+B''_{0,2} = 0 \ldots\ldots\ldots\ldots(63)h$$

$$B''_{2,0}-Z''_{2,0}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(62)h$$

and *mutatis mutandis*, the rest of the discussion is at page 305h, except that we have no systems of homogeneous equations in q_0 to satisfy, and, consequently, avoid the causes of failure which attend Mr. Jerrard's mode of effecting this last transformation. A general view of our method would seem to show that those causes of failure are always avoided by it. I shall push these considerations further in another paper.

Note.—Under the more general form of a *method of similar functions* I shall apply the analysis of p. 114m, to the resolution of differential and functional equations, &c.

Grecian Chambers, Devereux Court,
16*th December*, 1844.

ON THE EQUILIBRIUM OF ROOFS.

[*Mr. Rutherford.*]

To determine the conditions of equilibrium of any number of beams forming a roof in a vertical plane, symmetrical with respect to a vertical line through the highest point, and having weights, arising either from lateral beams or otherwise, placed on the ridge and shoulders.

Let DCBAB'C'D' be an equilibrated roof symmetrical with respect to the vertical line AK, and let

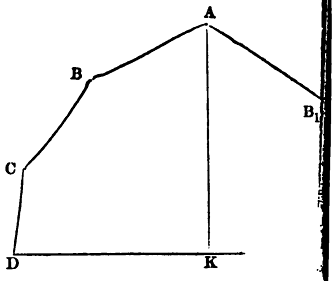
b, b_1 b_2, . . denote the lengths of the beams AB, BC, CD . .

a, a_1, a_2 . . , ... the angles which these beams respectively make with the horizon,

W, W_1, W_2, . . the weights of the several beams reckoning from the ridge,

A, A_1, A_2, . . the weights on the ridge and shoulders, and

H . . the horizontal thrust which, since the roof is in equilibrium, is constantly the *same throughout* the entire roof.



Then since the beams are considered uniform, the weights W, W_1, W_2,..may be regarded as weights suspended from the middle points of the several beams; and therefore the *vertical* pressure arising from the weight of the beam AB may be separated into two equal *vertical* pressures at the extremities A and B of the beam; hence the vertical pressure at A arising from the weight of the beam AB is equal to $\frac{1}{2}W$. For the same reason the vertical pressure at A arising from the upper beam on the opposite side of the roof is equal to $\frac{1}{2}W$; therefore the whole *vertical* pressure at A is $=A+\frac{1}{2}W+\frac{1}{2}W=A+W$. Now if P denote the pressure at A in direction BA; then the horizontal resolvent is $P\cos a$, which being equated to H, gives $P=H\sec a$, the pressure at A in direction BA; consequently the three pressures at the point A are, $H\sec a$ in direction BA; $H\sec a$ in direction B_1A; and $A+W$ in the direction of gravity; and since these pressures are, by hypothesis, in equilibrium, each pressure is proportional to the sine of the angle made by the directions of the other two; hence by Lami's principle

$$\frac{A+W}{H\sec a}=\frac{\sin(\pi-2a)}{\sin(\frac{1}{2}\pi+a)}=\frac{\sin 2a}{\cos a}=2\sin a;$$

$$\therefore \frac{A+W}{2H}=\tan a \dots\dots\dots\dots (1)$$

Again at the point B, the three pressures are $H\sec a$ in direction AB; $H\sec a_1$ in direction CB; and $A_1+\frac{1}{2}(W+W_1)$ in a vertical direction; hence

$$\frac{A_1+\frac{1}{2}(W+W_1)}{H\sec a}=\frac{\sin\{\pi-(a_1-a)\}}{\sin(\frac{1}{2}\pi-a)}=\frac{\sin(a_1-a)}{\cos a}=\sin a_1-\cos a_1\tan a;$$

$$\therefore \frac{2A_1+W+W_1}{2H}=\tan a_1-\tan a \dots\dots\dots\dots (2)$$

Adding equations (1) and (2) we obtain

$$\frac{A+2(A_1+W)+W_1}{2H}=\tan a_1 \dots\dots\dots\dots (3)$$

Similarly for the point C we find

$$\frac{2A_2+W_1+W_2}{2H}=\tan a_2-\tan a_1, \dots\dots\dots\dots (4)$$

which being added to the previous equation (3) gives

$$\frac{A+2(A_1+A_2+W+W_1)+W_2}{2H}=\tan a_2 \dots\dots\dots (5)$$

The same mode of reasoning is applicable throughout the entire roof, and we have therefore the following series of equations:

$$\left.\begin{aligned}\tan a &= \frac{A+W}{2H}\\ \tan a_1 &= \frac{A+2(A_1+W)+W_1}{2H}\\ \tan a_2 &= \frac{A+2(A_1+A_2+W+W_1)+W_2}{2H}\\ \vdots\ & \qquad\qquad \vdots\end{aligned}\right\} \dots\dots\dots\dots (6)$$

If s denote DK the half span of the roof; and $h=$ the height AK; then from the geometry of the figure we have

$$b\cos a+b_1\cos a_1+b_2\cos a_2+\ldots.=s\ldots\ldots\ldots\ldots\ldots(7)$$
$$b\sin a+b_1\sin a_1+b_2\sin a_1+\ldots.=h\ldots\ldots\ldots\ldots\ldots(8)$$

The following numerical example will serve for illustration.

Four uniform beams are symmetrically arranged in the form of a roof whose span is 24 feet; the beams at the ridge are each six feet in length, and the lower beams are each 10 feet: find the height of the roof, and the position of the beams.

From the equilibrium of the structure, and the geometry of the figure, we have, as in the preceding investigation,

$$\tan a=\frac{W}{2H}\ldots\ldots\ldots\ldots(1);\quad \tan a_1=\frac{2W+W_1}{2H}\ldots\ldots\ldots(2)$$
$$b\cos a+b_1\cos a_1=s\ldots..(3);\quad b\sin a+b_1\sin a_1=h\ldots\ldots\ldots(4)$$

Dividing (2) by (1) we get

$$\frac{\tan a_1}{\tan a}=\frac{2W+W_1}{W}=2+\frac{W_1}{W};$$

and if the upper and lower beams have the same weight per foot, the weights of the beams will be proportional to their lengths; hence

$$W : W_1 : : b : b_1,$$

and the previous equation becomes

$$\frac{\tan a_1}{\tan a}=2+\frac{b_1}{b}=2+\frac{10}{6}=\frac{11}{3};$$

consequently we have the two equations

$$3\tan a_1-11\tan a=0\ldots\ldots(5)\qquad 5\cos a_1+3\cos a=6\ldots\ldots..(6)$$

to determine a_1 and a. Eliminating either of the unknown quantities, as a_1, we obtain an equation of the fourth degree in a, viz.

$$28\cos^4 a-112\cos^3 a+88\cos^2 a+121\cos a-121=0\ldots\ldots..(7)$$

an equation having at least one positive root, and also one negative root. The resolution of eq. (7) will determine the value of a, and thence by (5) or (6) the value of a_1. The positions of the beams are then known, and consequently by (4) the height of the roof is determined.

ON THE REDUCTION OF CERTAIN INTEGRALS TO MORE SIMPLE FORMS.

[*The Rev. Brice Bronwin, Penistone, Wakefield.*]

By differencing and differentiating relative to a constant, some complex integrals may be reduced to others simple and obtainable, and at the same time expressed simply and elegantly. The following examples will illustrate this assertion, and point out the way of adding to their numbers.

Let $\Delta r=2$; taking the successive differences relative to r, we shall find

$$\left.\begin{aligned}\Delta^{2n}\sin(r-2n)x&=(-1)^n\,2^{2n}\sin rx(\sin x)^{2n}\\ \Delta^{2n}\cos(r-2n)x&=(-1)^n\,2^{2n}\cos rx(\sin x)^{2n}\\ \Delta^{2n-1}\sin(r-2n+1)x&=(-1)^n\,2^{2n-1}\cos rx(\sin x)^{2n-1}\\ \Delta^{2n-1}\cos(r-2n+1)x&=(-1)^n\,2^{2n-1}\sin rx(\sin x)^{2n-1}\end{aligned}\right\}\ldots\ldots.(a)$$

If therefore $f(x)$ be a function of x, we have from the first of (a)

$$\int f(x)dx \sin rx\,(\sin x)^{2n} = \frac{(-1)^n}{2^{2n}}\Delta^{2n}\int f(x)dx \sin(r-2n)x \ldots\ldots (A)$$

Let $f(x) = 1$, and also $= e^{ax}$; and make $r-2n=k$; (A) will give

$$\int dx \sin rx(\sin x)^{2n} = \frac{(-1)^{n+1}}{2^{2n}}\Delta^{2n}\left(\frac{\cos kx}{k}\right)\ldots\ldots\ldots (1)$$

$$\int e^{ax}dx \sin rx\,(\sin x)^{2n} = \frac{(-1)^n}{2^{2n}}\Delta^{2n}\left\{\frac{e^{ax}(a\sin kx - k\cos kx)}{a^2+k^2}\right\}\ldots\ldots (2)$$

since $\int e^{ax}dx \sin kx = \dfrac{e^{ax}(a\sin kx - k\cos kx)}{a^2+k^2}$. It is to be observed here, that e denotes the base of hyp. logs., and $\Delta k = \Delta r$.

We obtain from these, by differentiating relative to r and a,

$$\int x^{2p}\,dx \sin rx(\sin x)^{2n} = \frac{(-1)^{n-p+1}}{2^{2n}}\left(\frac{d}{dr}\right)^{2p}\Delta^{2n}\left(\frac{\cos kx}{k}\right)\ldots\ldots\ldots (3)$$

$$\int x^{2p-1}dx \cos rx(\sin x)^{2n} = \frac{(-1)^{n-p}}{2^{2n}}\left(\frac{d}{dr}\right)^{2p-1}\Delta^{2n}\left(\frac{\cos kx}{k}\right)\ldots\ldots\ldots (4)$$

$$\int e^{ax}x^{2p-1}dx\cos rx(\sin x)^{2n} = \frac{(-1)^{n-p+1}}{2^{2n}}\left(\frac{d}{da}\right)^{2p-1}\Delta^{2n}\left\{\frac{e^{ax}(a\sin kx - k\cos kx)}{a^2+k^2}\right\}$$
$$\ldots\ldots (5)$$

$$\int e^{ax}x^{p}\,dx \sin rx\,(\sin x)^{2n} = \frac{(-1)^n}{2^{2n}}\left(\frac{d}{da}\right)^{p}\Delta^{2n}\left\{\frac{e^{ax}(a\sin kx - k\cos kx)}{a^2+k^2}\right\}$$
$$\ldots\ldots (6)$$

There may be other forms of $f(x)$ for which we could integrate; and we might treat each of the other three formulæ of (a) in a similar manner. Thus from those formulæ alone we could obtain, and express in simple terms, various complex integrals. But my object is only to point out the way.

Again, $(\Delta r = 2)$ we find

$$\Delta\left(\frac{1}{\sin rx}\right) = -2\,\frac{\sin x\cos(r+1)x}{\sin rx\sin(r+2)x};$$

$$\Delta\left(\frac{1}{\cos rx}\right) = 2\,\frac{\sin x\sin(r+1)x}{\cos rx\cos(r+2)x};$$

$$\int\frac{dx\sin x\cos(r+1)x}{\sin rx\sin(r+2)x} = -\frac{1}{2}\Delta\int\frac{dx}{\sin rx} = -\frac{1}{2}\Delta\left\{\frac{1}{r}\log.\tan.\frac{rx}{2}\right\}\ldots (7)$$

$$\int\frac{dx\sin x\sin(r+1)x}{\cos rx\cos(r+2)x} = \frac{1}{2}\Delta\int\frac{dx}{\cos rx} = \frac{1}{2}\Delta\left\{\frac{1}{r}\log.\tan.\left(\frac{\pi}{4}+\frac{rx}{2}\right)\right\}\ldots (8)$$

In like manner, if $\Delta r = 1$;

$$\Delta\tan rx = \frac{\sin x}{\cos rx\cos(r+1)x};$$

$$\Delta^2\tan rx = \frac{2\sin(r+1)x\,(\sin x)^2}{\cos rx\cos(r+1)x\cos(r+2)x};$$

$$\int\frac{dx\sin x}{\cos rx\cos(r+1)x} = \Delta\int dx\tan rx = -\Delta\left\{\frac{1}{r}\log.\cos rx\right\}\quad (9)$$

$$\int\frac{dx\sin(r+1)x\,(\sin x)^2}{\cos rx\cos(r+1)x\cos(r+2)x} = -\frac{1}{2}\Delta^2\left\{\frac{1}{r}\log.\cos rx\right\}\ldots\ldots\ldots (10)$$

We may differentiate the last four integrals for r; but the results would be complicated. And we may treat $\cot rx$ as we have the tangent.

If now we make $\Delta' r = 2p$, and take the $2m$, and $2m - 1$ differences of the first of (a), and afterwards change r into $r - 2mp$, and $r - 2mp + p$; we find

$$\Delta'^{2m}\Delta^{2n}\sin(r-2n-2mp)x = (-1)^{n+m}2^{2n+2m}\sin rx(\sin x)^{2n}(\sin px)^{2m}$$

$$\Delta'^{2m-1}\Delta^{2n}\sin(r-2n-2mp+p)x = (-1)^{n+m-1}2^{2n+2m}\cos rx(\sin x)^{2n}(\sin px)^{2m-1} \quad \ldots.(b)$$

To abridge, put $r - 2n - mp = k$, $e^{ax}(a\sin kx - k\cos kx) = \mathrm{X}$; and we find from the first of (b), precisely as we found (1), (2), *etc.*, from the first of (a);

$$\int dx \sin rx\,(\sin x)^{2n}(\sin px)^{2m} = \frac{(-1)^{n+m-1}}{2^{2n+2m}}\Delta'^{2m}\Delta^{2n}\left(\frac{\cos kx}{k}\right) \ldots\ldots\ldots(11)$$

$$\int e^{ax}dx \sin rx(\sin x)^{2n}(\sin px)^{2m} = \frac{(-1)^{n+m}}{2^{2n+2m}}\Delta'^{2m}\Delta^{2n}\left(\frac{\mathrm{X}}{a^2+k^2}\right) \ldots\ldots\ldots(12)$$

$$\int x^{2t}\,dx \sin rx(\sin x)^{2n}(\sin px)^{2m} = \frac{(-1)^{n+m-t-1}}{2^{2n+2m}}\left(\frac{d}{dr}\right)^{2t}\Delta'^{2m}\Delta^{2n}\left(\frac{\cos kx}{k}\right)(13)$$

$$\int x^{2t-1}dx\cos rx(\sin x)^{2n}(\sin px)^{2m} = \frac{(-1)^{n+m-t}}{2^{2n+2m}}\left(\frac{d}{dr}\right)^{2t-1}\Delta'^{2m}\Delta^{2n}\left(\frac{\cos kx}{k}\right)(14)$$

We might have found others by differentiating (12) for r or a; and we may treat the 2nd of (b) as we have done the 1st. We might also find three other sets of formulæ similar to (b) from the three last of (a), and might treat them in like manner. It is obvious that by an extension of the process we might express with the same simplicity still more complicated integrals. In the last four examples it will be remembered that $\Delta k = \Delta r = 2$, $\Delta'' k = \Delta' r = 2p$.

The two next are easily proved by differentiation relative to the variable x.

$$\int dx \cos x(\sin x)^r = \frac{(\sin x)^{r+1}}{r+1}, \int dx \sin x(\cos x)^r = -\frac{(\cos x)^{r+1}}{r+1} \ldots.(c)$$

From these we derive by differencing and differentiating for r, $\Delta r = 2$,

$$\int dx(\cos x)^{2n+1}(\sin x)^r = (-1)^n\Delta^n\left\{\frac{(\sin x)^{r+1}}{r+1}\right\} \ldots\ldots\ldots\ldots(15)$$

$$\int dx(\sin x)^{2n+1}(\cos x)^r = (-1)^{n+1}\Delta^n\left\{\frac{(\cos x)^{r+1}}{r+1}\right\} \ldots\ldots\ldots(16)$$

$$\int dx \cos x(\sin x)^r(\log.\sin x)^p = \left(\frac{d}{dr}\right)^p\left\{\frac{(\sin x)^{r+1}}{r+1}\right\} \ldots\ldots\ldots\ldots(17)$$

$$\int dx \sin x(\cos x)^r(\log.\cos x)^p = -\left(\frac{d}{dr}\right)^p\left\{\frac{(\cos x)^{r+1}}{r+1}\right\} \ldots\ldots\ldots(18)$$

$$\int dx(\cos x)^{2n+1}(\sin x)^r(\log.\sin x)^p = (-1)^n\left(\frac{d}{dr}\right)^p\Delta^n\left\{\frac{(\sin x)^{r+1}}{r+1}\right\} \ldots.(19)$$

$$\int dx(\sin x)^{2n+1}(\cos x)^r(\log.\cos x)^p = (-1)^{n+1}\left(\frac{d}{dr}\right)^p\Delta^n\left\{\frac{(\cos x)^{r+1}}{r+1}\right\} \ldots(20)$$

Q Q

We might have made $\Delta r=1$ in the second of (c). And instead of th formulæ we might have employed $\tan x$, $\cot x$, or other functions. It also to be observed, that in all the preceding formulæ we might have u $\sin(rx+b)$, $\cos(rx+b)$, $\tan(rx+b)$, instead of $\sin rx$, $\cos rx$, $\tan r$ and might thus have obtained integrals more general.

Let $\Delta r=1$; then $\Delta\tan^{-1}(rx)=\tan^{-1}\left(\frac{x}{1+(1+r)rx^2}\right)$;

$\int dx\tan^{-1}(rx)=x\tan^{-1}(rx)-\frac{1}{2r}\log.(1+r^2x^2)=\mathrm{X}$ suppose; and

$$\int dx\tan^{-1}\left(\frac{x}{1+(1+r)rx^2}\right)=\Delta.\mathrm{X}\ldots\ldots\ldots\ldots(\text{!}$$

We might have multiplied by $f(x)dx$ instead of dx, provided we co integrate $\int f(x)\,dx\tan^{-1}(rx)$; and we might employ second or hig differences, which would give rise to very complicated functions.

Put D for $\frac{d}{dr}$; we may easily see that $(1+k\mathrm{D})e^{-rx}=(1-kx)e^{-rx}$

Repeating the operation m times, $(1+k\mathrm{D})^m e^{-rx}=(1-kx)^m e^{-rx}$.

Therefore $\int f(x)dx(1-kx)^m e^{-rx}=(1+k\mathrm{D})^m\int f(x)dxe^{-rx}\ldots\ldots($

Make $f(x)=x^a$; and since $\mathrm{D}^a e^{-rx}=(-1)^a x^a e^{-rx}$, $\int x^a dxe^{-rx}$:

$$(-1)^a\mathrm{D}^a\int dxe^{-rx}=(-1)^{a+1}\mathrm{D}^a\left(\frac{e^{-rx}}{r}\right);\ \text{(B) becomes}$$

$$\int e^{-rx}x^a\,dx\,(1-kx)^m=(-1)^{a+1}\mathrm{D}^a\,(1+k\mathrm{D})^m\left(\frac{e^{-rx}}{r}\right)\ldots\ldots\ldots\ldots($$

Again, make $f(x)=\sin px$ and $\cos px$; also $\int e^{-rx}dx\sin px=$ $\int e^{-rx}dx\cos px=\mathrm{X}_1$, these integrals being well known; and (B) give

$$\int e^{-rx}dx\sin px(1-kx)^m=(1+k\mathrm{D})^m\mathrm{X}\ldots\ldots\ldots\ldots($$

$$\int e^{-rx}dx\cos px(1-kx)^m=(1+k\mathrm{D})^m\mathrm{X}_1\ldots\ldots\ldots\ldots($$

$$\int e^{-rx}x^a\,dx\sin px(1-kx)^m=(-1)^a\,\mathrm{D}^a\,(1+k\mathrm{D})^m\mathrm{X}\ldots\ldots($$

$$\int e^{-rx}x^a\,dx\cos px(1-kx)^m=(-1)^a\,\mathrm{D}^a\,(1+k\mathrm{D})^m\mathrm{X}_1\ldots\ldots($$

The two last are obtained from those preceding by differentiating a ti for r.

In like manner, $(1+k\mathrm{D}^2)^m\sin rx=(1-kx^2)^m\sin rx$, $(1+k\mathrm{D}^2)^m\cos r$: $(1-kx^2)^m\cos rx$; and therefore

$$\int dx\sin rx(1-kx^2)^m=-(1+k\mathrm{D}^2)^m\left(\frac{\cos rx}{r}\right)\ldots\ldots\ldots\ldots($$

$$\int dx\cos rx(1-kx^2)^m=(1+k\,\mathrm{D}^2)^m\left(\frac{\sin rx}{r}\right)\ldots\ldots\ldots\ldots($$

$$\int e^{-ax}dx\sin rx\,(1-kx^2)^m=(1+k\mathrm{D}^2)^m\mathrm{X}\ldots\ldots\ldots\ldots($$

$$\int e^{-ax}dx\cos rx\,(1-kx^2)^m=(1+k\mathrm{D}^2)^m\mathrm{X}_1\ldots\ldots\ldots\ldots($$

where X and X_1 are as in (23), (24), *etc.*, r being changed into a, a *into* r. These last may be differentiated for r and a.

If $F(x)$ be a function developable by the powers of x,

$$F(D)e^{-rx} = F(-x)e^{-rx},\quad F(D^2)\sin rx = F(-x^2)\sin rx,$$
$$F(D^2)\cos rx = F(-x^2)\cos rx.$$ Therefore

$$\int dx\, F(-x)e^{-rx} = -F(D)\left(\frac{e^{-rx}}{r}\right) \qquad (31)$$

$$\int dx F(-x^2)\sin rx = -F(D^2)\left(\frac{\cos rx}{r}\right) \qquad (32)$$

$$\int dx\, F(-x^2)\cos rx = F(D^2)\left(\frac{\sin rx}{r}\right) \qquad (33)$$

$$\int dx\, F(-x)\sin px e^{-rx} = F(D)X. \qquad (34)$$

$$\int dx\, F(-x)\cos px e^{-rx} = F(D)X_1 \qquad (35)$$

where X and X_1 are as in (23), (24).

$$\int dx F(-x^2)\sin rx\, e^{-ax} = F(D^2)X \qquad (36)$$

$$\int dx\, F(-x^2)\cos rx\, e^{-ax} = F(D^2)X_1 \qquad (37)$$

where X and X_1 are as in (29), (30).

These may be differentiated for r, p, and a.

I have not thought it necessary to encumber this paper with the arbitrary constants which must be added to the integrals, as every one knows how to supply them.

DEVELOPEMENT OF POISSON'S METHOD OF FINDING THE RESULTANT OF TWO EQUAL FORCES.

[*Mr. James Anderson, Montrose.*]

Of the different methods of proving the parallelogram of forces, there is, perhaps, none more deserving of attention than that which Poisson has given in his "Traité de Mecanique." He first determines, by means of a functional equation, the resultant of two equal forces: from this he proceeds by means of a simple geometrical proof to the determination of the magnitude and direction of the equivalent of two rectangular forces; and then, in a manner exceedingly clear and explicit, advances to the general cases of two forces, whose directions make any angle whatever. It is the object of this paper to develop the method of finding the resultant of two equal forces, to consider it in some parts in a manner slightly different from that which he has pursued, and to shew that in one small particular his premises do not bear out the result.

Let P be the magnitude of each equal force, 2θ the angle which their directions contain, and R the resultant of the forces. There is no doubt about the direction of R, as there is no reason why it should incline to one of its components rather than to the other; its direction will therefore make an angle θ with those of each of the equal forces. Now P and θ being given, R must be of some determinate magnitude; in other words, R must be some determinable function of P and θ. It is also evident that, θ remaining the same, R must increase with P: thus if P should become nP, R would become nR; hence we may assume

$$R = P\phi\theta,$$

where $\phi\theta$ is some function of θ, which it is required to determine.

We might suppose one of the equal forces P to be compounded of two equal forces Q, each making some angle z with the direction of P; and therefore for the same reason we shall have

$$P = Q\phi z.$$

One of these forces will make with the direction of R the angle $\theta+z$, and the other (z being supposed less than θ) the angle $\theta-z$. If we suppose the other force P resolved in the same manner into two equal forces Q, each making an angle z with it, we shall have, instead of the two original forces, four equal forces, two of which make an angle $\theta+z$ with R, and the other two an angle $\theta-z$. By what we have already seen, the resultant of the one couple will be $Q\phi(\theta+z)$, and that of the other $Q\phi(\theta-z)$, both resultants directed according to R, and together having an equivalent equal to R.

Now it is admitted that when two forces act in the same direction, their resultant is equal to their sum, and hence

$$Q\phi(\theta+z)+Q\phi(\theta-z)=R=P\phi\theta=Q\phi\theta\phi z,$$

$$\text{or} \qquad \phi\theta\phi z=\phi(\theta+z)+\phi(\theta-z) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

the functional equation, from which the form of the function is to be determined.

In the Traité de Mecanique, at least in the second edition, Poisson merely observes, that $\phi\theta=2\cos a\theta$ will satisfy this equation, and asserts that this is the only function which will satisfy it. This is scarcely correct, as there are two distinct forms of functions which satisfy the equation, although it will appear, that, when constants are determined by means of known cases, they will lead, as they ought, to precisely the same results.

Developing by Taylor s Theorem

$$\phi\theta\phi z=2\left\{\phi\theta+\phi''\theta.\frac{z^2}{1.2}+\ldots\ldots+\phi^{2n}\theta.\frac{z^{2n}}{1.2..2n}+..\right\}$$

$$\therefore\ \phi z=2\left\{1+\frac{\phi''\theta}{\phi\theta}.\frac{z^2}{1.2}+\ldots\ldots+\frac{\phi^{2n}\theta}{\phi\theta}.\frac{z^{2n}}{1.2..2n}+..\right\}$$

Now since z is arbitrary, ϕz ought to be wholly independent of θ, and hence any coefficient $\frac{\phi^{2n}\theta}{\phi\theta}$ must be a constant. In like manner $\frac{\phi^{2n+2}\theta}{\phi\theta}$ must be a constant, so that $\frac{\phi^{2n+2}\theta}{\phi^{2n}\theta}$ must also be constant.

Let $\frac{\phi^{2n+2}\theta}{\phi^{2n}\theta}=a^2$; then denoting $\phi^{2n}\theta$ by y, we have $\frac{d^2y}{d\theta^2}=a^2y$, and solving this well known differential equation, we have

$$y=\phi^{2n}\theta=A_1e^{a\theta}+A'_1e^{-a\theta},\ \text{if } a^2 \text{ be positive,}$$

$$\text{and}\quad y=\phi^{2n}\theta=C_1\cos a\theta+C'_1\sin a\theta,\ \text{if it be negative.}$$

Thus no differential coefficient of $\phi\theta$ can consist of more than two terms, and any one may, for any thing that appears from the functional equation, consist of two, the constants of course for every differential coefficient being possibly different. We may hence make

$$\phi\theta=Ae^{a\theta}+A'e^{-a\theta},\ \text{ or }\ \phi\theta=C\cos a\theta+C'\sin a\theta,$$

these being the only, and the most general functions by which (1) can be satisfied. It may be observed that the former of these is the equation of the catenary.

It remains to determine the three constants, and this must be effected by means of our knowledge of three known cases. Now if the forces act in the same direction, that is, if $2\theta=0$, it is plain that the resultant will be 2P ;

hence $\phi\,(0) = 2$. Also if the forces act in opposite directions, that is, if $2\theta = \pi$, their resultant must be zero, and hence $\phi\left(\frac{\pi}{2}\right) = 0$. Again, if three equal forces are inclined to one another at angles of 120°, it is plain that their joint action will be nothing, since there is no reason why it should tend in any one direction rather than in any other; hence a force equal and opposite to one of these will be the equivalent of the other two, so that we have $\phi\left(\frac{\pi}{3}\right) = 1$. By means of these three cases, we will determine the constants in the expression

$$\phi\theta = C\cos a\theta + C'\sin a\theta.$$

If we take $\theta = 0$, then we get $\phi\,(0) = 2 = C$, and further

$$\text{if } \theta = \frac{\pi}{2}, \text{ then } \phi\left(\frac{\pi}{2}\right) = 0 = C\cos\frac{\pi a}{2} + C'\sin\frac{\pi a}{2};$$

$$\text{if } \theta = \frac{\pi}{3}, \text{ then } \phi\left(\frac{\pi}{3}\right) = 1 = C\cos\frac{\pi a}{3} + C'\sin\frac{\pi a}{3}.$$

Eliminating C and C′ it will be found, that there results

$$\sin\frac{\pi a}{2} = 2\sin\frac{\pi a}{2}\cos\frac{\pi a}{3} - 2\cos\frac{\pi a}{2}\sin\frac{\pi a}{3} = 2\sin\frac{\pi a}{6}.$$

Now $\sin\frac{\pi a}{2} = \sin\left(3.\frac{\pi a}{6}\right) = 3\sin\frac{\pi a}{6} - 4\sin^3\frac{\pi a}{6}$, and hence

$$3\sin\frac{\pi a}{6} - 4\sin^3\frac{\pi a}{6} = 2\sin\frac{\pi a}{6},$$

from which it follows that

$$\sin\frac{\pi a}{6} = 0, \text{ or } \sin\frac{\pi a}{6} = \pm\tfrac{1}{2}.$$

If $\sin\frac{\pi a}{6} = 0$, we shall find C′ infinite, so that we are justified in rejecting this solution. It follows, therefore, that $\sin\frac{\pi a}{6} = \pm\frac{1}{2}$, and that the arc $\frac{\pi a}{6}$ is either $(6n+1)\frac{\pi}{6}$, or $(6n+5)\frac{\pi}{6}$, where n is any integer, zero included. From this we find

$$a = 6n + 1, \text{ or } a = 6n + 5; \text{ and } C' = 0.$$

The expression now becomes

$$\phi\theta = 2\cos(6n+1)\theta, \text{ or } \phi\theta = 2\cos(6n+5)\theta;$$

an expression which suits the functional equation, and the three particular cases, and which yet is not correct, unless $n = 0$.

Taking the other functional form $\phi\theta = Ae^{a\theta} + A'e^{-a\theta}$, we shall have by means of the three known cases, the three following equations,

$$2 = A + A',$$

$$0 = Ae^{\frac{\pi a}{2}} + A'e^{-\frac{\pi a}{2}},$$

$$1 = Ae^{\frac{\pi a}{3}} + A'e^{-\frac{\pi a}{3}}.$$

From the second we find $A' = -Ae^{\pi a}$, and from this conjoined with the first, we find

$$A = \frac{2}{1-e^{\pi a}}, \text{ and } A' = \frac{-2e^{\pi a}}{1-e^{\pi a}}.$$

Substituting these values in the third, and reducing, there results

$$e^{\pi a} - 2e^{\frac{2\pi a}{3}} + 2e^{\frac{\pi a}{3}} - 1 = 0,$$

a reciprocal equation of an odd degree in respect to $e^{\frac{\pi a}{3}}$. Hence one solution is $e^{\frac{\pi a}{3}} = 1$. This value of a renders both A and A' infinite, and therefore we may reject it. Dividing by the factor $e^{\frac{\pi a}{3}} - 1$, we have

$$e^{\frac{2\pi a}{3}} - e^{\frac{\pi a}{3}} + 1 = 0.$$

If we divide x^3+1 by $x+1$, we get a function of x precisely the same as the left side of the former equation is of $e^{\frac{\pi a}{3}}$; and hence the two values of $e^{\frac{\pi a}{3}}$, will be the same as the two impossible roots of $x^3+1=0$; or in other words, the value of $e^{\frac{\pi a}{3}}$ is either of the impossible cube roots of -1.

Hence $e^{\pi a} = -1$, although its cube root $e^{\frac{\pi a}{3}}$ is not -1, an observation to which it is necessary to attend. From this value of $e^{\pi a}$ we find

$$A = 1, \text{ and } A' = 1.$$

Again, by the 7th chap. of De Morgan's Differential Calculus

$$e^{(2m+1)\pi\sqrt{-1}} = -1\text{; so that}$$

$$e^{\pi a} = e^{(2m+1)\pi\sqrt{-1}}, \text{ or } a = (2m+1)\sqrt{-1}.$$

But since $e^{\frac{\pi a}{3}}$ is not -1, it follows that $e^{\frac{2m+1}{3}\pi\sqrt{-1}}$ must not be -1, which it will be if $\frac{2m+1}{3}$ be an odd integer. That $\frac{2m+1}{3}$ may not be an odd integer, it is necessary that $2m$ be of the form $6n$ or $6n+4$; and hence $a = (6n+1)\sqrt{-1}$, or $a = (6n+5)\sqrt{-1}$; and the expression now becomes

$$\phi\theta = e^{(6n+1)\theta\sqrt{-1}} + e^{-(6n+1)\theta\sqrt{-1}},$$

$$\text{or, } \phi\theta = e^{(6n+5)\theta\sqrt{-1}} + e^{-(6n+5)\theta\sqrt{-1}};$$

and by the chap. of De Morgan's Diff. Cal. already referred to, these are the same as

$$\phi\theta = 2\cos(6n+1)\theta, \text{ or } \phi\theta = 2\cos(6n+5)\theta,$$

the same result as we arrived at by the discussion of the other functions.

To ascertain that n should be nothing, we may observe that $\phi\theta$ ought never to be nothing, unless $\theta = \frac{\pi}{2}$, or more generally a multiple of $\frac{\pi}{2}$; but by the formula $\phi\theta$ will be 0, when $\theta = \frac{\pi}{2(6n+1)}$, which cannot be the case, consistently with what has been observed, unless $n = 0$.

HORNER ON ALGEBRAIC TRANSFORMATION.

(Continued from p. 142.)

27. We come now to the subject of numerical *synthesis* and *analysis* in their more enlarged acceptation; and as the conditions of the General Theorem do not bind us to any particular law of variation, we shall for the most part take the increments of the root to mean simply the digits of which it is composed, rated at their proper value in the scale of notation.

The order in which we take up the digits is optional, at least in synthetic operations. One of them, or a larger portion of the root, if convenient, being used as a constant multiplier throughout the first transformation, the subsequent process may be regulated by the general law.

Ex. 1. As a familiar illustration let it be required to find the cube of 835271.

First Variety. Assuming the coefficients of x^3 only, and commencing with the final digit: then the multipliers m are used p times entire, and then diminished from the right.

1	0	0	0
	1	1	1
$(m^p =) 1^3$	1	1	1 $= 1^3$
	71	72	35791
	..	504	......
	72	5113	357911 $= 71^3$
$7,1^2,7$	271	2401	195446
	....	686	......
$2,7,1^1,27,2$	343	97723	19902511 $= 271^3$
	527	11226	146426615
	....	28065	....
$5,27^1$, &c.	5613	29285323	146446517511 $= 5271^3$
	352	204065	4373220969
	......	122439	....
$3,5,2^1$, &c.	40813	1457740323	43878656207511 $= 35271^3$
	835	2627439	5827060242584
	......	7006504	....
$8,3,5^1$	875813	728382530323	582749902914607511 $= 835271^3$

In ordinary calculation, the first two lines of this work might have been dispensed with, as well as the valuation of the progressive cubics. These

particulars are introduced solely for the sake of illustration. Respecting the mode of operating, the law of the addends to the first column is sufficiently plain; a portion of that addend, not exceeding two figures, and those the latest introduced into it, is used to multiply the sum beneath it. The partial products appear in their places in the corresponding portion of the second column; and the amount of that portion being multiplied by one digit only of the same multiplier, and that the latest introduced into it, the product is carried forward to the final column. We pass on to

Second Variety, commencing with the first digit, and founded, as is usual in the cube root, on the coefficients of $(x_8 + 8)^3$.

	24 .	192 . .	
	3	729	512 . . .
			52787 . . .
3^2	243	19929	10395875 . . .
	35	7395	418435208 . . .
		12325	146499676183 . . .
			2093030424511
$3_{,}5^2,5$	2465	2079175	
	352	125010	5827499029146075ll.
		50004	
			the cube required.
$3_{,}5_{,}2^1,52,2$	25002	209217604	
	527	501094	
		1753829	
$5_{,}2_{,}7^1$, &c.	250547	20928525169	
	271	17540187	
		2505741	
$2_{,}7_{,}1^1$, &c.	2505741	2093030424511	

The multipliers in this and all the succeeding examples are diminished from the *left*.

28. But if we reflect on the distinctive characters of *evolution*, which were discussed on a former occasion, it will appear that in this last variety the example is treated by the method which is the best adapted of any to the purpose of *analysis*. By commencing at the top of the numerical scale we ensure the increasing stability of the divisor and coefficients in general. Also, in assuming $fx + f(x_m + r_m)$ as the base of the work, we accomplish the same thing as is effected in equations in general by the method of *limits* already explained: viz. the possession of a tolerably trustworthy divisor in the first instance. We shall, therefore, consult the general convenience of the reader by giving such an interpretation of the general theorem as will accord with this procedure.

Multiply the first coefficient by the last n *digits of the root, or by all the digits if fewer than* n, *and add the product to the second coefficient: to the third coefficient, add the product of the former sum by the last* n—1 *of those digits, or by all of them if fewer than* n—1: *and so on.*

The product which is finally carried forward to the closing column will be either an addend or a subtrahend, according as the problem is synthetic or analytic.

Ex. 2. Extract the fourth root of 1517108809906561.

	24.	216..	864...	1517108809906561(6241 *
	2	484	44168	1296
2^4	242.	22084	908168..	2211088
	24	4888	4534112	1816336
		9776	9068224	
				3917520990
24^3	2444.	2267056	962577344	3850309376
	241	49362	930614884	
		49872	232653721	972116146561
		24681		972116146561
241^2	24681	232653721	972116146561	......

29. In multiplying, it will be observed that I begin with the left hand figure† of the multiplier. The utility of this arrangement will be felt in forming the additions, and still more when the inferior addends are to be rejected in contracting the work. Nor need the stationing of the superior addend occasion any perplexity. We have only to observe how the first multiplying digit we use is situated with regard to the coefficient above it in the *first* column; for *its successive products must advance continually one place further to the right.*

Or thus, more generally:—

If the new coefficient in any column advances p figures to the right of the old one, and is to be multiplied by m digits, the first addend to the next old coefficient must begin $p+1-(m-1)$, *that is* $p-m+2$, *places to the right of it. If* $m \gt 2$, *this will mean* $p+m-2$ *places to the left of it.*

30. By making p negative, we hence obtain a similar rule for commencing the *contractions*: viz.—*if p figures are cut off from any coefficient, and that coefficient is to receive m addends,* $p-m+2$ *figures must be cut off from the next coefficient to the left, and so on, in retrogradation. When* $m \gt 2$ *this will mean that* $p+m-2$ *digits or some representative marks must be added.*

The due extent of the first addend being thus secured, and designated by vertical lines, the remaining $m-1$ places of each new coefficient may be cut off singly by points, and partially neglected, while the remaining addends are found and arranged after the manner usual in contracted multiplication.

At each subsequent course of contractions, the vertical line must be drawn at a uniform interval to the left of all the points.

* A small portion of the foot of the page is cut away, but whether when originally sent to the Royal Society, or on subsequently shortening the paper, I have no means of discovering. Nor is there any clue to the cancelled passage. It is, however, probably not very material.
T. S. D.

† The oriental nations do so universally still, except when they have imported the European method—as in the *Schools* of Turkey and India. Our method is on many accounts the least convenient in its applications; and even we are *virtually* compelled to adopt it in the contracted multiplication of decimals; and we actually adopt the same principle in duodecimal operations, as well as in our algebraic multiplication. Why then not adopt it in *toto,* and make it the rule from the outset of a child's entering upon arithmetic? We should find no disadvantage in any stage, but gain some important ones (especially unity of method) by its adoption. T. S. D.

Ex. 3. Find the root of $x^5+2x^4+3x^3+4x^2+5x-321=0$.

Initial Solution, by the method of Art. (23),

	1	2	3	4	5	— 321
1^5........	1	3	6	10	15	— 306
2^4	1	5	16	42	99	— 207
3^3........	1	8	40	162	423	+ 216

Hence the root begins with 2. We complete the transformation by Art. (24),

	1	5	16	42	99	—207
2^3........	1	7	30	102	201	
2^2........	1	9	48	150		
2^1........	1	11	59			
1	1	12				

Ulterior Solution. The root of the reduced equation

$$x_2^5+12x_2^4+59x_2^3+150x_2^2+201x_2-207=0$$

is *greater* than $\frac{207}{423}$ or $\cdot 5$; and it is *less* than $\frac{207}{201+75+15}$ or $\cdot 7$. '
fore, *(as on opposite page.)*

Hence the root is 2·638605803324, correct in the 12th decimal. former communication* the last figure is brought out too great, in quence of a slight error committed in the first column at the commen of the contractions.

31. In re-proposing this problem, I wish to invite comparison wi method by *pure transformées*, already before the public. It will, I appear that if the moulded precision of the latter method renders it impossible to make a slip either in the arrangement or contraction work: yet the new method has important advantages peculiar to itse employs fewer addends, and these are determined by multiplication s and not by the combined mental use of multiplication and addition; ı any diversity of sign ever occur in a set of addends.

For the rest, it is clear that every thing which was advanced to el the universal adaptation of the prior method to equations, applies a with equal force to this. For in the investigation, N, M, L, *et* interchangeable with the universal expressions f, Df, D^2f, *etc.*, by irrational and transcendental quantities are reducible to the regula braic form.

In fact, as we have seen, the two methods differ solely in this re that the new process transforms by means of the radical digits alone; in the other, each digit is succeeded by n zero-increments. By usin of these, many diversities of solution may be composed: but it is su to have mentioned them.

32. Before we quit this part of the subject, a feature in the th equations deserves to be noticed, which harmonizes in a peculiar ı with the principles of transformation by division, from which the investigation proceeds. It is this—that *depression* and *continuo proximation are one and the same.* As soon as any root of the e

* Phil. Trans. 1819. The same paper is reprinted in the Appendix to the Ladi for 1838, without this correction; as I was not then aware of it, not having at th *reprinting* that paper, the present one in my possession. T.

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                            12.        59..          150              201                207   (·638605803324
                             6          756          39936            1139616            18897696
6^5 ..........             126.        6656          189936           3149616            180230400000
                            63          7938         4493694          1422718722         139304119743
                                        3969         2246847          711359361          40926280257
6,3^4 ..........           1323.      748949         23711978|7       4643470658|1       38031030028
                            638        83208         5024561|04       87164285|82        2895250229
6,3,8^3 ......           13|8,6,8       4160|4       251228|05        23243809|55        2867504760
                           |6 3 8|6     1109|44      66994|15         475387875|3|47     27745469
6,3,8,6^2   , , 1,4|51             837|4,2,6|84     290547|6,1|94     2353102|0 8        23904795
                                       87|06         2790|27          176482|6 5         3840674
                                        4|35          744|07          4779174|60|0 8     3824781
                                        1|16           55|80          1769|58            15893
                                         |9          294|13,7|76      14|75              14343
                                     9,3,0|09            |74          478095|8|93        1550
                                                         |5                1|4 7         1434
                                                    2,9,4|93                |2 3         116
                                                                      478097|5 9         95
                                                                                         21
                                                                                         19
```

is determined accurately, or to a sufficient extent, the absolute term is actually or virtually cancelled; and the remaining coefficients constitute a depressed formula whence the other real roots may be elicited.

When the method of pure transformées is employed, the depressed formula is already fit for solution, being a pure function of the remaining roots, severally diminished by the known root, omitting its last increment. If the transformations have been of the mixed kind, and one zero-increment be applied to rectify the new absolute term, the roots are the entire difference of the unknown roots and the known one. Or, if r be the root last determined, the increment $-r$ may be used, which will restore the original roots.

Various inconveniences, however, attend the practical application of these principles to the purpose of *ulterior* solution, arising particularly from the great number of digits in these residual coefficients, and the necessity of a complicated and tardy management of the contractions. On these accounts, I believe it will, for the most part, be found preferable to evolve each root separately from the data of the *initial solution*. It is, therefore, chiefly in reference to this stage of approximation, that the preceding remarks are practically useful; as by their aid it effects with superior convenience, the purpose of the *method of divisors*.

Ex. What are the roots of $x^4+2x^3-22x^2+7x+42=0$?

	1	2	-22	7	$+42$	
1^4	1	3	-19	-12	$+30$	
2^3	1	5	-9	-30	$+0$	$\therefore$ 2 is a root.
2^2	1	7	5	-25		$\therefore$ 3 is a root.
3^1	1	10	25	0		
-1^1	1	9	7			
-2	1	7				

Consequently the roots are 2, 3, and the roots of $x^2+7x+7=0$, or, 2, 3, and $-\frac{1}{2}(7\pm\sqrt{-11})$. Here the regular increment of 1 is superseded after the proof of the root 2, by 0; and after the proof of 3, first by -3, because the signs being all positive, shewed that all the positive roots were found, and twice by 0 to complete the pure transformée which contains the other roots.

MATHEMATICAL NOTES.

I.

Upon two lines AB, AB′ *meeting in* A, *two points* C *and* C′ *are respectively taken such that* AC. AC′= BC. BC′, *find the curve which is constantly touched by a straight line passing through* C *and* C′.

Taking the lines as axes of co-ordinates and A as origin, let $AB=a$, $AB'=a'$; also $AC=b$, and $AC'=b'$: then we have

$$bb'=(a-b)(a'-b'),\ \text{or}\ aa'=a'b+ab'\ldots\ldots\ldots\ldots(1)$$

The equation also of the line CC′ is

$$\frac{x}{b'}+\frac{y}{b}=1\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1, 2) we get by differentiation with respect to b and b',

$$a'+a\frac{db'}{db}=0,\ \text{and}\ \frac{y}{b^2}+\frac{x}{b'^2}\cdot\frac{db'}{db}=0;$$

And by elimination of $\frac{db'}{db}$ from these

$$\frac{a^{\frac{1}{2}}y^{\frac{1}{2}}}{b}=\frac{a'^{\frac{1}{2}}x^{\frac{1}{2}}}{b'}\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

If now we put the values of $\frac{1}{b}$ and $\frac{1}{b'}$, obtained from (2, 3) in (1), we shall have

$$\left(\frac{y}{a}\right)^{\frac{1}{2}} + \left(\frac{x}{a'}\right)^{\frac{1}{2}} = 1,$$

the well known equation of a parabola, obtained by a correspondent in No. 3 of the Mathematician, page 143, which may also be seen more briefly educed in O'Brien's Co-ordinate Geometry. γ.

II.

[*Mr. John Laws, Newcastle.*]

Given the three angles of any plane triangle and a side, to find the radii of the inscribed and escribed circles.

If from a point within or without the triangle perpendiculars p_1, p_2, p_3, be let fall on the sides respectively opposite to the angles A, B, C, we shall have (*Hamilton's Analytical Geometry*, page 45,)

$$\begin{aligned} p_3 \sin C + p_2 \sin B \pm p_1 \sin A &= a \sin B \sin C \\ &= b \sin A \sin C \\ &= c \sin A \sin B, \end{aligned}$$

the upper or lower sign being taken according as the point is within or without the triangle.

Now if the point be equidistant from the sides, it becomes the centre of the circle which touches the sides internally or externally. Hence if r, r_1, r_2, r_3, be the radii of the inscribed and escribed circles respectively, we have

$$r = \frac{a \sin B \sin C}{\sin A + \sin B + \sin C} = \frac{b \sin A \sin C}{\sin A + \sin B + \sin C} = \frac{c \sin A \sin B}{\sin A + \sin B + \sin C}$$

$$r_1 = \frac{a \sin B \sin C}{\sin B + \sin C - \sin A} = \frac{b \sin A \sin C}{\sin B + \sin C - \sin A} = \frac{c \sin A \sin B}{\sin B + \sin C - \sin A}$$

$$r_2 = \frac{a \sin B \sin C}{\sin A + \sin C - \sin B} = \frac{b \sin A \sin C}{\sin A + \sin C - \sin B} = \frac{c \sin A \sin B}{\sin A + \sin C - \sin B}$$

$$r_3 = \frac{a \sin B \sin C}{\sin A + \sin B - \sin C} = \frac{b \sin A \sin C}{\sin A + \sin B - \sin C} = \frac{c \sin A \sin B}{\sin A + \sin B - \sin C}.$$

These values may also be otherwise expressed, viz.:

$$r = \frac{a \sin \frac{1}{2} B \sin \frac{1}{2} C}{\cos \frac{1}{2} A} = \frac{b \sin \frac{1}{2} A \sin \frac{1}{2} C}{\cos \frac{1}{2} B} = \frac{c \sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} C}$$

$$r_1 = \frac{a \cos \frac{1}{2} B \cos \frac{1}{2} C}{\cos \frac{1}{2} A} = \frac{b \sin \frac{1}{2} A \cos \frac{1}{2} C}{\sin \frac{1}{2} B} = \frac{c \sin \frac{1}{2} A \cos \frac{1}{2} B}{\sin \frac{1}{2} C}$$

$$r_2 = \frac{a \sin \frac{1}{2} B \cos \frac{1}{2} C}{\sin \frac{1}{2} A} = \frac{b \cos \frac{1}{2} A \cos \frac{1}{2} C}{\cos \frac{1}{2} B} = \frac{c \cos \frac{1}{2} A \sin \frac{1}{2} B}{\sin \frac{1}{2} C}$$

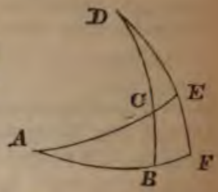
$$r_3 = \frac{a \cos \frac{1}{2} B \sin \frac{1}{2} C}{\sin \frac{1}{2} A} = \frac{b \cos \frac{1}{2} A \sin \frac{1}{2} C}{\sin \frac{1}{2} B} = \frac{c \cos \frac{1}{2} A \cos \frac{1}{2} B}{\cos \frac{1}{2} C}.$$

These are deduced from the preceding by means of the well known expressions:

$$\begin{aligned} \sin A + \sin B + \sin C &= 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C \\ \sin B + \sin C - \sin A &= 4 \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C \cos \tfrac{1}{2} A \\ \sin A + \sin C - \sin B &= 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} C \cos \tfrac{1}{2} B \\ \sin A + \sin B - \sin C &= 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \cos \tfrac{1}{2} C. \end{aligned}$$

FORMULÆ FOR RIGHT-ANGLED SPHERICAL TRIANGLES DEDUCED IN SUCCESSION.

(From a Correspondent.)



Let ABC be any right-angled spherical triangle, B being the right angle, and let DCE be the complemental triangle. Then in any spherical triangle

$$\cos B = \frac{\cos AC - \cos AB \cos BC}{\sin AB \sin BC}.$$

$$\begin{array}{lll}
B = 90^\circ, \text{ gives} & \cos AC - \cos AB \cos BC = 0, & \\
\text{or,} & \cos AC = \cos AB \cos BC \ldots\ldots\ldots\ldots\ldots\ldots & (1) \\
\text{From } (1) \ldots\ldots & \cos DC = \cos CE \cos ED, & \\
\text{or,} & \sin BC = \sin AC \sin A \ldots\ldots\ldots\ldots\ldots\ldots & (2) \\
\text{From } (2) \ldots\ldots & \sin DE = \sin DC \sin C, & \\
\text{or,} & \cos A = \cos BC \sin C \ldots\ldots\ldots\ldots\ldots\ldots & (3) \\
\text{From } (3) \ldots\ldots & \cos C = \cos DE \sin D, & \\
\text{or,} & \cos C = \sin A \cos AB \ldots\ldots\ldots\ldots\ldots\ldots & (4) \\
\text{From } (4) \ldots\ldots & \cos D = \sin C \cos CE, & \\
\text{or,} & \sin AB = \sin C \sin AC \ldots\ldots\ldots\ldots\ldots\ldots & (5) \\
\text{From } (5) \ldots\ldots & \sin CE = \sin D \sin DC, & \\
\text{or,} & \cos AC = \cos AB \cos BC \ldots\ldots\ldots\ldots\ldots\ldots & (1)
\end{array}$$

Multiply together equations (3) and (4),

$$\cos A \cos C = \cos AB \cos BC \sin A \sin C.$$

Divide by $\sin A \sin C$, then

$$\cot A \cot C = \cos AB \cos BC = \cos AC,$$

$$\begin{array}{lll}
\text{or,} & \cos AC = \cot A \cot C \ldots\ldots\ldots\ldots\ldots\ldots & (6) \\
\text{From } (6) \ldots\ldots & \cos DC = \cot C \cot D, & \\
\text{or,} & \sin BC = \cot C \tan AB \ldots\ldots\ldots\ldots\ldots\ldots & (7) \\
\text{From } (7) \ldots\ldots & \sin DE = \cot D \tan CE, & \\
\text{or,} & \cos A = \tan AB \cot AC \ldots\ldots\ldots\ldots\ldots\ldots & (8) \\
\text{From } (8) \ldots\ldots & \cos C = \tan CE \cot DC, & \\
\text{or,} & \cos C = \cot AC \tan BC \ldots\ldots\ldots\ldots\ldots\ldots & (9) \\
\text{From } (9) \ldots\ldots & \cos D = \cot DC \tan DE, & \\
\text{or,} & \sin AB = \tan BC \cot A \ldots\ldots\ldots\ldots\ldots\ldots & (10) \\
\text{From } (10) \ldots\ldots & \sin CE = \tan DE \cot C, & \\
\text{or,} & \cos AC = \cot A \cot C \ldots\ldots\ldots\ldots\ldots\ldots & (6)
\end{array}$$

Greenwich, May 2, 1845. J. R.

SOLUTIONS OF MATHEMATICAL EXERCISES.

XXXII.—*Pen-and-Ink.*

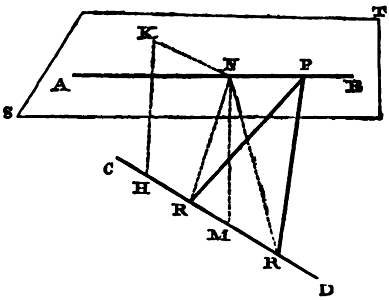
Two straight lines being given in space, and a point in one of them: it is required to draw through the given point a line to meet the other given line, and make equal angles with them both.

[First Solution.—*Mr. Hugh Godfray, Val Plaisant, Jersey.*]

Any two lines in space may be conceived to have come to the position which they occupy, by revolving about their common perpendicular, in planes at right angles to that perpendicular. Now, when they are in the same plane, it is evident that the line which joins two points in them equidistant from this axis, will be equally inclined to them both; and that

the revolution itself will not in any way affect the *equality* of these angles, into whatever position they may be brought by means of this revolution—though the *actual magnitude* of those angles may be varied to any extent between certain definable limits. Whence the following

Construction. Let AB, CD be the two given lines, and P a point in AB. Draw the plane ST through AB parallel to CD; and through any point H in CD draw HK perpendicular to ST; and likewise KN, NM parallel respectively to CD, KH. Lastly set off MR, MR′, each equal NP, on different sides of M upon the line CD; and join PR, PR′: these lines will each be equally inclined to AB and CD.



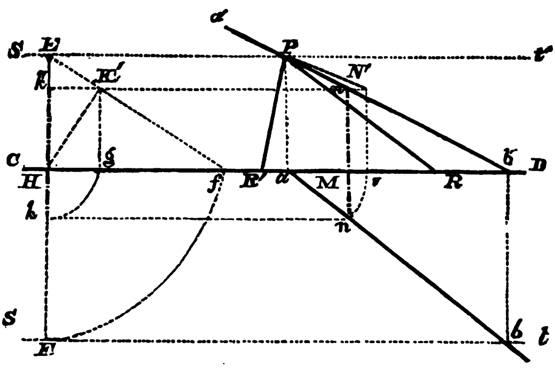
Demonstration. For, join NR and PM. Then it is obvious that MN is perpendicular to both AB and CD; and the two right-angled triangles NMR, PNM, have the sides about their right angles equal each to each; and hence PM is equal to NR. From this, it follows that the two triangles NRP, PMR have all the sides of the one equal to all the sides of the other, each to each; and hence that their corresponding angles NPR, PRM are also equal; or PR makes equal angles with the given lines AB, CD.

In the same manner it may be proved that PR′ makes equal angles with AB and CD.

[SECOND SOLUTION.—*By the same Gentleman.*]

The construction by means of *Descriptive Geometry* may be very simply effected as follows; and for convenience of reference, the letters in the two figures are made to correspond.

Let the vertical plane be that which passes through CD and P, and let CD be the ground line; and let ab, $a'b'$ be the projections of AB; also let the traces of AB be found in the usual manner, P being one of them and b the other, as indicated by the *épure*.



Through P and b draw the parallels to CD, which will, obviously, be the traces of the plane ST through AB parallel to CD.

Take any convenient point H in CD, and draw the perpendicular EHF; then it is clear that the perpendicular HK from this point to the plane ST will be projected on HE, HF. The general construction for finding the projections of the point K fails in this case; but we may easily effect it as follows.

Conceive the triangle EHF to be revolved about EH, till it coincides with the vertical plane, and takes the position EH*f*, and EF coincides with E*f*. Draw HK′ perpendicular to E*f*; then K′*k*′, K′*g* being drawn perpendicular to HE, HD are the distances of the point K from the vertical and horizontal planes of projection. Whence making H*k* equal to H*g*, *k* and *k*′ are the horizontal and vertical projections of K.

Next, draw *kn*, *k*′*n*′ parallel to CD, meeting the projections *ab*, *a*′*b*′ of AB in *n*, *n*′; and the points *n*, *n*′ being the projections of N, the line *n*′ *n* is perpendicular to CD. Also since the line MN is parallel to HK, this perpendicular determines the point M on CD; and we thus have the projections of the common perpendicular to AB and CD.

To find the distance PN, make $av = an$, and draw *v*N′ parallel to M*n*′, meeting *k*′*n*′ in N′: then PN′ = PN, as is shewn amongst the elementary propositions of the science.

Lastly, on CD set off MR, MR′ each equal to PN′, and join PR, PR′; which are, obviously, the lines required.

By taking, as we have done, one of the lines CD for the ground line, and the plane through P for the vertical plane of projection, the construction is much simplified; and we get at once the true length of PR and its inclination PRC to the given lines.*

Mr. Bills, of Hawton, also favoured us with a solution, the same in principle, as that of Mr. Godfray; and Mr. Weddle, of Newcastle, with one effected by means of the right cylinder, and likewise another by the method of co-ordinates.

* The *Descriptive Geometry*, the object of which is to solve by constructions made in one plane, problems which relate to the three dimensions of space, is but little known in England; and with the exception of the *Prospectus to the "Mathematician,"* I have not seen it mentioned in any English mathematical treatise. In France it is one of the *essentials* in the examination of the Naval and Military Cadets.—H.G.

XXXIII.—*By* ±.

Let three parabolas be escribed to the three sides of a triangle, and have their principal axes in the lines bisecting the exterior angles; then, if a, b, c, represent the sides of the triangle, s half their sum, and p_1, p_2, p_3, the semiparameters of the three parabolas, it is required to shew that

$$p_1 p_2 p_3 = \frac{(a-b)(b-c)(c-a)s^3}{abc}.$$

[FIRST SOLUTION.—*Mr. G. W. Hearn, Royal Military College, Sandhurst.*]

Let AX bisect the exterior angle BAQ of the triangle ABC, and let

$$y^2 = 2p(x-m) \dots\dots\dots\dots (1)$$

be the equation of the escribed parabola PRVQ; where $m = \text{AV}$, the distance between the vertex of the parabola and the angular point A, and P, R, Q are the points of contact of the escribed parabola with the sides of the triangle. Let the angle BAX or XAQ $= a$, and transform equation (1) so that AQ may be the axis of x, and AB that of y; then the equation of the parabola is

$$(y-x)^2 \sin^2 a = 2p\{(x+y)\cos a - m\} \dots\dots\dots\dots (1')$$

In this make one of the variables $= 0$, and find the condition for equal roots, which will be

$$p \cos^2 a = 2m \sin^2 a \dots\dots\dots\dots (2)$$

This relation subsisting, both AB and CAQ will be tangents to the parabola.

The equation to CB is (making, as usual, $AC = b$, $AB = c$, $CB = a$,)

$$\frac{y}{c} - \frac{x}{b} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Eliminating y between (1′) and (3), and finding the condition for equal roots, we have

$$p(c+b)^2 \cos^2 a = 4bc(c-b)\sin^2 a \cos a + 2m\,(c-b)^2 \sin^2 a,$$

which insures CB being tangential.

Eliminating m between this and (2), gives

$$p = (c-b)\,\frac{\sin^2 a}{\cos a} = (c-b)\,\frac{\cos^2\frac{1}{2}A}{\sin\frac{1}{2}A} = \frac{(c-b)\,s\,(s-a)}{\sqrt{\{bc(s-b)(s-c)\}}}.$$

Since $\cos a$ is necessarily positive we see that for the parabola to be escribed as we have supposed, it is necessary that $c > b$.

Assuming that $c > b$ and $b > a$, we have, by multiplying the three expressions for p_1, p_2, p_3,

$$p_1 p_2 p_3 = \frac{(c-b)(b-a)(c-a)s^3}{abc}, \text{ or } \frac{(a-b)(b-c)(c-a)s^3}{abc}.$$

Cor. By eliminating p instead of m, we find $m = \dfrac{c-b}{2} \sin \dfrac{A}{2}$:

$$\text{whence } 8m_1 m_2 m_3 = (c-b)(b-a)(c-a).\frac{(s-a)(s-b)(s-c)}{abc};$$

$$\therefore \frac{8m_1 m_2 m_3}{p_1 p_2 p_3} = \frac{(s-a)(s-b)(s-c)}{s^3};$$

$$\therefore \text{area } \Delta = s^2 \left(\frac{2m_1}{p_1}.\frac{2m_2}{p_2}.\frac{2m_3}{p_3}\right)^{\frac{1}{2}}.$$

Or, if D_1, D_2, D_3 be the respective distances of the vertices of the escribed parabolæ from the angles A, B, C, and p_1, p_2, p_3 the distances of the vertices from the foci,

$$\text{area } \Delta = s^2 \left(\frac{D_1 D_2 D_3}{p_1 p_2 p_3}\right)^{\frac{1}{2}}.$$

[SECOND SOLUTION.—*Mr. S. Bills, Hawton; and similarly by Mr. W. Marr, Edinburgh.*]

Let ABC be a plane triangle, AD being the line bisecting the exterior angle at A, and meeting BC produced in D. Let DA produced be the principal axis, and V the vertex of a parabola which touches the side CA in Q, and the sides BC, BA produced in P, R. Draw the ordinates PM, RN, and put $AV = x$, $DV = y$. By known properties of the triangle and parabola, we readily obtain the following relations:

$$\angle RAN = 90° - \frac{A}{2},\ \angle PDN = \frac{B-C}{2},\ AD = \frac{2bc\sin\frac{1}{2}A}{b-c};\text{ whence}$$

$$y - x = AD = \frac{2bc\sin\frac{1}{2}A}{b-c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$2p_1 x = 4x^2 \cot^2\tfrac{1}{2}A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$2p_1 y = 4y^2.\left(\frac{b-c}{b+c}\right)^2 \cot^2\tfrac{1}{2}A \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From (2), (3) we find

$$x = \tfrac{1}{4} p_1 \tan^2 \tfrac{1}{2} \mathrm{A}, \quad y = \tfrac{1}{2} p_1 \tan^2 \tfrac{1}{2} \mathrm{A} \left(\frac{b+c}{b-c}\right)^2.$$

Substituting these values in (1), and reducing, we obtain

$$p_1 = (b-c)\cos\tfrac{1}{2}\mathrm{A}\cot\tfrac{1}{2}\mathrm{A} = \frac{s(s-a)}{bc}\left\{\frac{bc}{(s-b)(s-c)}\right\}^{\frac{1}{2}}(b-c) \ldots (4)$$

Similarly, $p_2 = (a-c)\cos\tfrac{1}{2}\mathrm{B}\cot\tfrac{1}{2}\mathrm{B} = \dfrac{s(s-b)}{ac}\left\{\dfrac{ac}{(s-a)(s-c)}\right\}^{\frac{1}{2}}(a-c) \ldots (5)$

and, $p_3 = (a-b)\cos\tfrac{1}{2}\mathrm{C}\cot\tfrac{1}{2}\mathrm{C} = \dfrac{s(s-c)}{ab}\left\{\dfrac{ab}{(s-a)(s-b)}\right\}^{\frac{1}{2}}(a-b) \ldots (6)$

Multiplying together (4), (5), (6), we get

$$p_1 p_2 p_3 = \frac{(a-b)(a-c)(b-c)s^3}{abc}.$$

Correct solutions were also received from Messrs. John Laws, Thomas Weddle, R. H. Wright, and the Proposer.

XXXIV.—*Mr. Thomas Weddle, Newcastle.*

Sum each of the series,

$$(\sec\theta + 1)\ (\sec\frac{\theta}{2} + 1)\ (\sec\frac{\theta}{2^2} + 1) \ldots\ldots (\sec\frac{\theta}{2^{n-1}} + 1),$$

$$(2\cos\theta + 1)(2\cos\frac{\theta}{3} + 1)\ (2\cos\frac{\theta}{3^2} + 1) \ldots\ldots (2\cos\frac{\theta}{3^{n-1}} + 1),$$

$$(2\cos\theta - 1)(2\cos\frac{\theta}{3} - 1)\ (2\cos\frac{\theta}{3^2} - 1) \ldots\ldots (2\cos\frac{\theta}{3^{n-1}} - 1).$$

[SOLUTION.—*By the Proposer.*]

Let S_n, s_n, and σ_n be the sums of the first, second, and third series respectively.

Since, $1 + \sec\omega = \dfrac{1+\cos\omega}{\cos\omega} = \dfrac{2\cos^2\frac{1}{2}\omega}{\cos\omega} = \dfrac{2\cos^2\frac{1}{2}\omega.\sin\frac{1}{2}\omega}{\cos\omega.\sin\frac{1}{2}\omega}$

$$= \frac{\cos\frac{1}{2}\omega.\sin\omega}{\sin\frac{1}{2}\omega.\cos\omega} = \frac{\cot\frac{1}{2}\omega}{\cot\omega}.$$

Hence $\quad \dfrac{\mathrm{S}_{n+1}}{\mathrm{S}_n} = \sec\dfrac{\theta}{2^n} + 1 = \dfrac{\cot\dfrac{\theta}{2^{n+1}}}{\cot\dfrac{\theta}{2^n}}$;

$\therefore\ \mathrm{S}_n = \mathrm{C}.\cot\dfrac{\theta}{2^n}$; but $\mathrm{S}_1 = \mathrm{C}\cot\frac{1}{2}\theta = \sec\theta + 1 = \dfrac{\cot\frac{1}{2}\theta}{\cot\theta}$ $\therefore\ \mathrm{C} = \tan\theta$

$$\therefore \mathrm{S}_n = \tan\theta.\cot\frac{\theta}{2^n}.$$

Again, $2\cos\omega + 1 = \dfrac{2\cos\omega.\sin\omega + \sin\omega}{\sin\omega} = \dfrac{\sin2\omega + \sin\omega}{\sin\omega} = \dfrac{2\sin\frac{3}{2}\omega.\cos\frac{1}{2}}{2\sin\frac{1}{2}\omega\cos\frac{1}{2}}$

$= \dfrac{\sin\frac{3}{2}\omega}{\sin\frac{1}{2}\omega}$; hence $\dfrac{s_{n+1}}{s_n} = 2\cos\dfrac{\theta}{3^n} + 1 = \dfrac{\sin\dfrac{1}{3^{n-}}.\dfrac{\theta}{2}}{\sin\dfrac{1}{3^n}.\dfrac{\theta}{2}}$ $\quad\therefore s_n = \dfrac{\mathrm{C}}{\sin\dfrac{1}{3^{n-1}}}.$

Now, $s_1 = \dfrac{C}{\sin\frac{\theta}{2}} = 2\cos\theta + 1 = \dfrac{\sin\frac{3}{2}\theta}{\sin\frac{\theta}{2}}, \quad \therefore C = \sin\frac{3}{2}\theta.$

Hence, $s_n = \dfrac{\sin\frac{3}{2}\theta}{\sin\frac{1}{3^{n-1}}.\frac{\theta}{2}} = \sin\frac{3}{2}\theta.\ \text{cosec}\ \frac{1}{3^{n-1}}.\frac{\theta}{2}.$

In like manner, from $2\cos\omega - 1 = \dfrac{\cos\frac{3}{2}\omega}{\cos\frac{1}{2}\omega}$, we find,

$$\sigma_n = \frac{\cos\frac{3}{2}\theta}{\cos\frac{1}{3^{n-1}}.\frac{\theta}{2}} = \cos\tfrac{3}{2}\theta.\ \sec\frac{1}{3^{n-1}}.\frac{\theta}{2}.$$

Good Solutions were sent by Messrs. G. W. Hearn, R. M. College, and William Marr, Edinburgh.

XXXV.—*Mr. Matthew Collins, Limerick.*

How many terms of the squares of the numbers 1, 2, 3, 4, *etc.*, must be added that the sum may be a rational square number?

[First Solution.—*Mr. Geo. W. Hearn, R. M. College, Sandhurst.*]

The sum of the series $1^2+2^2+3^2+\ldots.n^2$ is $\dfrac{n(n+1)(2n+1)}{1.2.3}$, and it is required to assign a value of n such as to make this a perfect square. Let $2n+1 = m$; then $n = \dfrac{m-1}{2}$, $n+1 = \dfrac{m+1}{2}$, and the sum$=\dfrac{m(m^2-1)}{4\times 6}$; so that we have to make $\dfrac{m(m^2-1)}{6}$ a square. Let $m^2-1=6z^2$, or $m^2-6z^2=1$; then the general solution of this is easily found by developing $\sqrt{6}$ in a continued fraction. If any of the values of m is an odd square, all the conditions are satisfied. We find $m = 49$; hence $n = 24$, and the sum of 24 terms $= 4\times 25\times 49 = 70^2$.

[Second Solution.—*Mr. John Laws, Newcastle-on-Tyne.*]

The sum of the squares of the numbers 1, 2, 3, 4, 5, 6,..........n, or $1^2+2^2+3^2+4^2+\ldots\ldots.n^2$ is $\dfrac{n(n+1)(2n+1)}{6}$.

Let $\dfrac{n}{6} = p^2$, $\therefore n+1 = 6p^2+1 =$ a square, and $2n+1 = 12p^2+1 =$ a square.

Assume $12p^2+1=(ap-1)^2=a^2p^2-2ap+1$, $\therefore p = \dfrac{2a}{a^2-12}$.

Let $a = 4$, then $p = 2$, $6p^2+1 = 25$, $n = 24$, and the sum of 24 terms of the series

$$= 4\,.\,25\,.\,49 = 2^2.\,5^2.\,7^2 = 70^2.$$

Messrs. S. Bills, Hawton; W. Marr, Edinburgh; Thomas Weddle, Newcastle; and the proposer; sent solutions to this exercise.

XXXVI.—*Mr. J. W. Elliott, Greatham.*

Find the equation of a plane meeting three given straight lines in sp
and making equal angles with them.

[FIRST SOLUTION.—*Mr. G. W. Hearn, and similarly by Mr. W. Marr, Edinbu*

Let $a_1\, b_1\, c_1$ be the direction cosines of one of the given lines; $a_2\, l$
$a_3\, b_3\, c_3$ those of the other lines; $x\, y\, z$ those of a perpendicular to
required plane. Then the cosines of the angles which this perpendic
makes with the three given lines are respectively

$$a_1x + b_1y + c_1z$$
$$a_2x + b_2y + c_2z$$
$$a_3x + b_3y + c_3z,$$

which are to be mutually equal.

Hence $$a_1 - a_2 + (b_1 - b_2)\frac{y}{x} + (c_1 - c_2)\frac{z}{x} = 0,$$

$$a_1 - a_3 + (b_1 - b_3)\frac{y}{x} + (c_1 - c_3)\frac{z}{x} = 0,$$

$$(a_1 - a_2)(c_1 - c_3) - (a_1 - a_3)(c_1 - c_2)$$
$$+\{(b_1 - b_2)(c_1 - c_3) - (b_1 - b_3)(c_1 - c_2)\}\frac{y}{x} = 0;$$

$$\therefore \frac{y}{x} = \frac{(a_1c_3 + a_2c_1 + a_3c_2) - (a_2c_3 + a_1c_2 + a_3c_1)}{(b_2c_3 + b_1c_2 + b_3c_1) - (b_1c_3 + b_2c_1 + b_3c_2)}.$$

Similarly, $$\frac{z}{x} = \frac{(a_1b_3 + a_2b_1 + a_3b_2) - (a_2b_3 + a_1b_2 + a_3b_1)}{(c_2b_3 + c_1b_2 + c_3b_1) - (c_1b_3 + c_2b_1 + c_3b_2)};$$

from which, if X Y Z be the co-ordinates of any point in the required p
its equation will be

$$\begin{aligned}&\mathrm{X}\ \{(b_3c_1 + b_1c_2 + b_2c_3) - (b_1c_3 + b_2c_1 + b_3c_2)\}\\ &+\mathrm{Y}\ \{(a_1c_3 + a_2c_1 + a_3c_2) - (a_3c_1 + a_1c_2 + a_2c_3)\}\\ &+\mathrm{Z}\ \{(a_1b_2 + a_2b_3 + a_3b_1) - (a_2b_1 + a_3b_2 + a_1b_3)\}\\ &= \text{an arbitrary constant.}\end{aligned}$$

[SECOND SOLUTION.—*Mr. Thomas Weddle, Newcastle.*]

Let, $$\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma} \ldots\ldots\ldots\ldots$$

$$\frac{x}{\cos\alpha'} = \frac{y}{\cos\beta'} = \frac{z}{\cos\gamma'} \ldots\ldots\ldots\ldots$$

$$\frac{x}{\cos\alpha''} = \frac{y}{\cos\beta''} = \frac{z}{\cos\gamma''} \ldots\ldots\ldots\ldots$$

be the equations of three lines drawn through the origin parallel to the g
lines, and

$$\frac{x}{\cos\alpha_1} = \frac{y}{\cos\beta_1} = \frac{z}{\cos\gamma_1} \ldots\ldots\ldots\ldots$$

those of a line making equal angles with them; let the cosine of ea
these equal angles be denoted by c; then will

$$\cos\alpha.\cos\alpha_1 + \cos\beta.\cos\beta_1 + \cos\gamma.\cos\gamma_1 = c \ldots\ldots\ldots$$
$$\cos\alpha'.\cos\alpha_1 + \cos\beta'.\cos\beta_1 + \cos\gamma'.\cos\gamma_1 = c \ldots\ldots\ldots$$
and, $$\cos\alpha''.\cos\alpha_1 + \cos\beta''.\cos\beta_1 + \cos\gamma''.\cos\gamma_1 = c \ldots\ldots\ldots$$

Eliminate $\cos a_1$ from (5, 6), from (6, 7), and from (5, 7), we thus find

$$(\cos a' \cos\beta - \cos a \cos\beta')\cos\beta_1 + (\cos a'\cos\gamma - \cos a \cos\gamma')\cos\gamma_1 = c(\cos a' - \cos a) \quad \ldots.(8)$$

$$(\cos a'' \cos\beta' - \cos a' \cos\beta'')\cos\beta_1 + (\cos a''\cos\gamma' - \cos a'\cos\gamma'')\cos\gamma_1 = c(\cos a'' - \cos a') \quad \ldots.(9)$$

$$(\cos a \cos\beta'' - \cos a'' \cos\beta)\cos\beta_1 + (\cos a \cos\gamma'' - \cos a'' \cos\gamma)\cos\gamma_1 = c(\cos a - \cos a'') \quad \ldots.(10)$$

Hence, if we put

$$A = \cos\beta'\cos\gamma - \cos\beta\cos\gamma' + \cos\beta\cos\gamma'' - \cos\beta''\cos\gamma + \cos\beta''\cos\gamma' - \cos\beta'\cos\gamma'' \quad \ldots.(11),$$

$$B = \cos a\cos\gamma' - \cos a'\cos\gamma + \cos a''\cos\gamma - \cos a\cos\gamma'' + \cos a'\cos\gamma'' - \cos a''\cos\gamma \quad \ldots.(12)$$

and, $$C = \cos a'\cos\beta - \cos a\cos\beta' + \cos a.\cos\beta'' - \cos a''\cos\beta + \cos a''\cos\beta' - \cos a'\cos\beta'' \quad \ldots.(13)$$

and, add (8, 9, 10), we have

$$C\cos\beta_1 - B\cos\gamma_1 = 0.$$

A similar process would give

$$C\cos a_1 - A\cos\gamma_1 = 0,$$

and, $$B\cos a_1 - A\cos\beta_1 = 0.$$

Hence, $\dfrac{\cos a_1}{A} = \dfrac{\cos\beta_1}{B} = \dfrac{\cos\gamma_1}{C}$, and therefore (4) becomes

$\dfrac{x}{A} = \dfrac{y}{B} = \dfrac{z}{C}$; the equation of a plane perpendicular to this is, $Ax + By + Cx = P$, and this is evidently the plane required. The values of A, B, C are given by (11, 12, 13), and P is an arbitrary constant.

Mr. Weddle sent another neat solution to this exercise.

[Third Solution —*Mr. J. W. Elliott, the Proposer.*]

Let AB, A′B′, A″B″ be the three straight lines; and take A for the origin of rectangular co-ordinates, AB for axis of z; then the equations of those lines, are respectively

$$\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right\} \qquad \left.\begin{array}{l} x = az + a \\ y = bz + \beta \end{array}\right\} \qquad \left.\begin{array}{l} x = a'z + a' \\ y = b'z + \beta' \end{array}\right\} \ldots\ldots(1)$$

Let the equation of the plane be

$$z + Ax + By + D = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

then it will be necessary to determine A, B, in terms of the constants in (1). By Waud's Anal. Geom. p. 219, the sine of the angle which AB makes with the plane is

$$\frac{1}{(1 + A^2 + B^2)^{\frac{1}{2}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and the sines of the angles which A′B′, A″B″ make with it, are respectively

$$\frac{1 + Aa + Bb}{\sqrt{(1 + A^2 + B^2)(1 + a^2 + b^2)}}, \quad \frac{1 + Aa' + Bb'}{\sqrt{(1 + A^2 + B^2)(1 + a'^2 + b'^2)}} \ldots(4, 5)$$

Equating (3) with (4, 5), and reducing, we find

$$A = \frac{b'\{(1+a^2+b^2)^{\frac{1}{2}}-1\}-b\{(1+a'^2+b'^2)^{\frac{1}{2}}-1\}}{ab'-a'b},$$

$$B = \frac{a'\{(1+a^2+b^2)^{\frac{1}{2}}-1\}-a\{(1+a'^2+b'^2)^{\frac{1}{2}}-1\}}{a'b-ab'}.$$

These values of A, B substituted in (2) will give the equation of the plane required; and as D remains indeterminate, we infer, that every plane drawn parallel to that just found, will make equal angles with the three straight lines.

XXXVII.—*By ϕ.*

The six vertices of two triangles about a conic section lie also in a conic section.

[FIRST SOLUTION.—*Mr. G. W. Hearn, Royal Military College.*]

The intersections of the sides of the two triangles will be the angular points of a hexagon circumscribed about the conic section, and therefore having the necessary and sufficient property that its diagonals intersect in the same point.

Let the sides of one triangle be denoted by the equations $u_1=0$, $u_2=0$, $u_3=0$, and those of the other triangle by $v_1=0$, $v_2=0$, $v_3=0$, so that u_1v_1, u_2v_2, u_3v_3, may apply to opposite sides of the before-mentioned hexagon.

Now suppose the origin taken at the point of mutual intersection of the three diagonals, and that each of the above linear equations is put under the form $ax+by-1=0$. Then h, k, l, being three determinable constants, the conditions that the three diagonals may pass through the origin, are

$$\left.\begin{array}{l} u_1-v_2=h(v_1-u_2) \\ u_2-v_3=k(v_2-u_3) \\ u_3-v_1=l\,(v_3-u_1) \end{array}\right\}\dots\dots\dots\dots\dots\dots (a)$$

$$\text{or, } u_1(1+lhk) = h(1+k)\,v_1+(1-hk)\,v_2+h(lk-1)v_3$$
$$u_2(1+lhk) = k(lh-1)\,v_1+k\,(1+l)\,v_2+(1-lk)v_3$$
$$u_3(1+lhk) = (1-lh)\,v_1+l(hk-1)\,v_2+l\,(1+h))v_3.$$

Now the equation to a conic section circumscribing the triangle $u_1u_2u_3$ may be written under the form

$$u_1u_2+Pu_1u_3+Qu_2u_3=0\dots\dots\dots\dots(1)$$

since this is of the second order, and is satisfied by any two of the three conditions $u_1=0$, $u_2=0$, $u_3=0$. Provided then we can determine P and Q in such a manner as to reduce the preceding to the form

$$v_1v_2+P'v_1v_3+Q'v_2v_3=0\dots\dots\dots\dots(2)$$

the object will be accomplished, since this latter is a conic section circumscribing $v_1v_2v_3$; and it will therefore follow that we have found a conic section circumscribing both triangles. Putting therefore the values of $u_1u_2u_3$ in terms of $v_1v_2v_3$ in the equation (1), and making the coefficients of $v_1^2\,v_2^2\,v_3^2$ separately $=0$, we have the following conditions,

$$Ph\,(1+k)-Qk\,(1-lh)=h\,k\,(1+k)$$
$$Pl\,(1-hk)+Qlk\,(1+l)=\;k\,(1+l)$$
$$Plh\,(1+h)-Q\,l\,(1+h)=-h\,(1-lk);$$

but since the second multiplied by h subtracted from the first multiplied by l re-produces the third, these conditions are equivalent to only two independent conditions, and the values of P and Q are

$$P = k(1 + l) \div l(1 + h) \text{ and } Q = h(1 + k) \div l(1 + h),$$

and therefore

$$l(1 + h) u_1 u_2 + k(1 + l) u_1 u_3 + h(1 + k) u_2 u_3 = 0,$$

is the equation to the conic section circumscribing both triangles.

hkl are immediately deducible from the conditions of identity (a). Thus if $u_1 = a_1 x + b_1 y - 1$, $v_1 = \alpha_1 x + \beta_1 y - 1$, *etc.*; then

$$h = \frac{a_1 - \alpha_2}{\alpha_1 - a_2} = \frac{b_1 - \beta_2}{\beta_1 - b_2}, \text{ etc.}$$

This theorem is a deduction from Pascal's properties of the hexagramme, and is easily reducible to the following elegant property of converging triangles. Let there be two convergent triangles, and let two other triangles circumscribe both of them, these two latter will not in general be convergent, but on repeating the process, that is circumscribing two other triangles about both the triangles last described, the two triangles so circumscribed will always be convergent.

[SECOND SOLUTION.—*Mr. Thomas Weddle.*]

$$\text{Let } A = 0,\ B = 0,\ C = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

be the equations of the three sides of one of the triangles. The equation

$$A^2 + M^2B^2 + N^2C^2 + 2PAB + 2QAC + 2RBC = 0. \ldots\ldots(2)$$

denotes a conic section; and if the line $C = 0$ touches it, the resulting values of x and y, and therefore of A and B, must be equal; this requires that $A^2 + M^2B^2 + 2BAP$ should be a complete square, $\therefore M^2 = P^2$, or $P = M$. In like manner, if the other sides touch the curve, we shall have $Q = N$ and $R = \pm MN$; if these values be substituted in (2), and the upper sign of R be taken, we shall get merely the equation of a straight line; take therefore the lower sign, and we have

$$A^2 + M^2B^2 + N^2C^2 + 2MAB + 2NAC - 2MNBC = 0 \ldots\ldots\ldots(3)$$

for the equation of every conic section touching the sides of the triangle (1).

Now, by giving proper values to b and c, any straight line may be denoted by $A = bB + cC$; let therefore the equations of the sides of the other triangle be

$$A = bB + cC \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$
$$A = b_1B + c_1C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$
$$A = b_2B + c_2C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Eliminate A from (3) by means of (4),

$$\therefore (M + b)^2B^2 + 2\{bc + Mc + Nb - MN\}BC + (N + c)^2C^2 = 0.$$

Since (4) touches (3), this must be a complete square, therefore

$$\pm(M + b)(N + c) = bc + Mc + Nb - MN.$$

The upper sign must be rejected, for then either $M = 0$ or $N = 0$, neither of which is admissible; take therefore the lower sign,

$$\therefore bc + Mc + Nb = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

In the same way, we get from (5) and (6)

$$b_1c_1 + Mc_1 + Nb_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$\text{and} \quad b_2c_2 + Mc_2 + Nb_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Multiply (7) by $\frac{1}{bc}$, (8) by $\frac{\lambda}{b_1c_1}$, and (9) by $\frac{\mu}{b_2c_2}$, add and equate the coefficients of M and N to zero, and reduce

$$\left.\begin{array}{r} 1+\lambda+\mu=0 \\ b_1b_2+\lambda bb_2+\mu bb_1=0 \\ c_1c_2+\lambda cc_2+\mu cc_1=0 \end{array}\right\} \dots\dots\dots\dots\dots\dots,(10)$$

The elimination of λ and μ from the equations (10), will give the relation among $b\,b_1b_2\;c\,c_1c_2$, which must subsist that the two triangles whose sides are denoted by (1) and (4, 5, 6) may circumscribe the same conic section.

Again, P and Q being arbitrary constants, the equation of any conic section circumscribing the triangle whose sides are (4, 5, 6) may be denoted by

$$(b_1B+c_1C-A)(b_2B+c_2C-A)+P\,(bB+cC-A)(b_2B+c_2C-A)$$
$$+Q\,(bB+cC-A)(b_1B+c_1C-A)=0$$

If this curve also circumscribe the triangle (1), it must satisfy $B=0$ and $C=0$, $A=0$ and $C=0$, and $A=0$ and $B=0$; these give the relations

$$\left.\begin{array}{r} 1+P+Q=0 \\ b_1b_2+bb_2P+bb_1Q=0 \\ \text{and}\quad c_1c_2+cc_2P+cc_1Q=0 \end{array}\right\} \dots\dots\dots\dots\dots\dots(11)$$

and the elimination of P and Q, in the usual manner, will give the relation which must subsist in order that the triangles (1) and (4, 5, 6) may be circumscribed by the same conic section; but (11) differing from (10) only in having P and Q instead of λ and μ, it is obvious that this relation *is* satisfied, and consequently that the same conic section circumscribes both triangles.

[THIRD SOLUTION.—*Mr. Samuel Bills, Hawton.*]

Let ABC, *abc* (the student will readily sketch the figure) be two triangles circumscribed to the same conic section, so that the tangents AC, BC may intersect *cb* in c_1 and b_1; and *ac*, *ab* may intersect AB in A_1 and B_1. Then (*Mathematician*, page 252,) the anharmonic ratios of $AA_1\;B_1B$ will be equal to those of $c_1c\;bb_1$. Hence, if *a*A, *a*B, C*c*, C*b* be joined, the radiants *a*A, aA_1, *a*B, aB_1, and Cc_1, C*c*, C*b*, Cb_1, will contain angles which will have equal anharmonic ratios. It therefore follows (*Mathematician*, page 250,) that the six points A, B, C, *a*, *b*, *c*, lie in a conic section.

Mr. Laws sent a good solution on the principle of poles and polars.

XXXVIII.—*By W. F., Durham.*

(1.) If from any point P in the plane of a quadrilateral figure ABCD, lines be drawn to the four angular points, then (the diagonals being drawn) there exists the following relation between the triangles

$$APC.PBD=APD.BPC\pm PDC.PAB,$$

the upper or lower sign being taken according as the point is *without* or *within* the quadrilateral.

(2.) If from both extremities of any line PQ in space, lines be drawn to the four angular points of a quadrilateral ABCD in space, then (the

diagonals being drawn) there exists the following relation between the pyramids,

$$APQC.PQDB = APQD.BCQP \pm PQDC.PQAB,$$

the upper or lower sign being taken according as the line PQ is *without* or *within* the quadrilateral.

[First Solution.—*Mr. G. W. Hearn, Royal Military College.*]

Denote the angles APB, BPC, CPD, APD, by α, β, γ, δ : then expressing the area of each triangle by the product of its sides into half the sine of the contained angle, it will be seen that when the point P is within the quadrilateral, the property enunciated reduces to

$$\sin(\alpha+\beta)\sin(\alpha+\delta) = \sin\beta\sin\delta - \sin\alpha\sin\gamma \ldots\ldots\ldots\ldots (a)$$
$$\text{where} \qquad \alpha+\beta+\gamma+\delta = 2\pi.$$

Since $\sin\gamma = \sin\{2\pi - (\alpha+\beta+\delta)\} = -\sin(\alpha+\beta+\delta)$, the second member of (a) becomes

$$\begin{aligned}\sin\beta\sin\delta + \sin\alpha\sin(\alpha+\beta+\delta) &= \sin\{(\alpha+\beta)-\alpha\}\sin\delta + \sin\alpha\sin\{(\alpha+\beta)+\delta\}\\ &= \sin(\alpha+\beta)\cos\alpha\sin\delta - \cos(\alpha+\beta)\sin\alpha\sin\delta\\ &\quad + \sin(\alpha+\beta)\cos\delta\sin\alpha + \cos(\alpha+\beta)\sin\alpha\sin\delta\\ &= \sin(\alpha+\beta)\sin(\alpha+\delta).\end{aligned}$$

The identity therefore of the two members of (a) is established.

When the point is without the quadrilateral, we have one of the angles as $\gamma = \alpha+\beta+\delta$, and the relation is

$$\sin(\alpha+\beta)\sin(\alpha+\delta) = \sin\beta\sin\delta + \sin\alpha\sin\gamma,$$

which may be proved in the same way.

Next (for the theorem in space,) let PQ produced cut the plane of the quadrilateral in R ; then if i be the angle of inclination which PRQ makes with this plane, we have

$$\begin{aligned}\text{pyramid ARCP} &= \tfrac{1}{3}\Delta\, \text{ARC.PR} \sin i\\ \ldots\ldots\ \text{ARCQ} &= \tfrac{1}{3}\Delta\, \text{ARC.QR} \sin i\\ \therefore\ \ldots\ldots\ \text{ACPQ} &= \tfrac{1}{3}\Delta\, \text{ARC(PR} \pm \text{QR)} \sin i\\ &= \tfrac{1}{3}\Delta\, \text{ARC.PQ} \sin i.\end{aligned}$$

Hence the second property follows from multiplying the equation relatively to the point R by $(\frac{1}{3}\text{PQ}\sin i)^2$, and is therefore also true.

[Second Solution.—*Mr. R. H. Wright,* London ; and similarly by the Proposer.*]

Let P be the origin of rectangular co-ordinates and without the quadrilateral, PB the positive axis of x, and the points A, B, C, D, respectively, (x_1y_1), (x_2o), (x_3y_3), (x_4y_4).

Now the expression for the area of a triangle in terms of the co-ordinates of its three angular points $(x_1y_1,\ x_2y_2,\ x_3y_3)$ in reference to rectangular axes is *(Young's Analytical Geometry)*,

$$\text{Area} = \tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\}.$$

Hence we readily get the following areas:

$$\Delta\,\text{APC} = \tfrac{1}{2}(x_1y_3 - x_3y_1), \qquad \Delta\,\text{PBD} = -\frac{x_2y_4}{2},$$

$$\Delta\,\text{APD} = \tfrac{1}{2}(x_1y_4 - x_4y_1), \qquad \Delta\,\text{BPC} = -\frac{x_2y_3}{2},$$

$$\Delta\,\text{PDC} = \tfrac{1}{2}(x_4y_3 - x_3y_4), \qquad \Delta\,\text{PAB} = -\frac{x_2y_1}{2};$$

* This gentleman is the author of "A Supplement to Elementary Algebra."

$$\therefore\quad \text{APC.PBD} = \tfrac{1}{4}(x_2x_3y_1y_4 - x_1x_2y_3y_4)\ldots\ldots\ldots\ldots(1)$$

$$\text{And, APD.BPC} + \text{PDC.PAB} = \tfrac{1}{4}(x_2x_4y_1y_3 - x_1x_2y_3y_4 + x_2x_3y_1y_4 - x_2x_4y_1y_3)$$

$$= \tfrac{1}{4}(x_2x_3y_1y_4 - x_1x_2y_3y_4)\ldots\ldots\ldots\ldots(2)$$

When the point P, then, is without the quadrilateral, we have

$$\text{APC.PBD} = \text{APD.BPC} + \text{PDC.PAB}.$$

In a similar way it may be shewn, that when the point P is within the figure,

$$\text{APC.PBD} = \text{APD.PBC} - \text{PDC.PAB}.$$

For the theorem in space, let P be the origin and PQ the axis of z. Denote also the points A, B, C, D, P, Q, thus:

$$(x_1y_1z_1),\ (x_2y_2z_2),\ (x_3y_3z_3),\ (x_4y_4z_4),\ (ooo),\ (ooz_6).$$

Now the content of a pyramid in terms of the co-ordinates of three of its angular points, the fourth being the origin, is (*Garnier Geo. Anal.* page 336.)

$$\text{Content} = \frac{1}{6}\left\{x_1(y_2z_3 - y_3z_2) + x_2(y_3z_1 - y_1z_3) + x_3(y_1z_2 - y_2z_1)\right\}.$$

Hence the following expressions for the respective pyramids:

$$\text{APQC} = \frac{z_6}{6}(x_3y_1 - x_1y_3), \qquad \text{PQDB} = \frac{z_6}{6}(x_4y_2 - x_2y_4),$$

$$\text{APQD} = \frac{z_6}{6}(x_4y_1 - x_1y_4), \qquad \text{BCQP} = \frac{z_6}{6}(x_3y_2 - x_2y_3),$$

$$\text{PQDC} = \frac{z_6}{6}(x_4y_3 - x_3y_4), \qquad \text{PQAB} = \frac{z_6}{6}(x_2y_1 - x_1y_2).$$

It is readily seen that

$$\text{APQC.PQDB} = \text{APQD.BCQP} + \text{PQDC.PQAB}.$$

The second part of the theorem in space is proved in a similar way.

A good solution was received from Mr. William Marr, of Edinburgh.

XL.—*Mr. Fenwick.*

Prove that if the three lines

$$\left.\begin{aligned} y &= a_1z + a' \\ x &= \beta_1z + \beta' \end{aligned}\right\}\ldots(1) \quad \left.\begin{aligned} y &= a_2z + a'' \\ x &= \beta_2z + \beta'' \end{aligned}\right\}\ldots(2) \quad \left.\begin{aligned} y &= a_3z + a''' \\ x &= \beta_3z + \beta''' \end{aligned}\right\}\ldots(3)$$

be mutually conjugate to one another (the perpendicular case included), there exists the relation

$$\frac{1}{a_1\beta_3} + \frac{1}{a_2\beta_1} + \frac{1}{a_3\beta_2} = \frac{1}{\beta_1a_3} + \frac{1}{\beta_2a_1} + \frac{1}{\beta_3a_2}.$$

[First Solution.—*Mr. Geo. W. Hearn, R. M. College, Sandhurst.*]

Let $$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 = 1,$$

represent a surface of the second order referred to its principal axes.

Then the equation to a plane conjugate to a diameter through $x'y'z'$ is

$$\text{A}xx' + \text{B}yy' + \text{C}zz' = 0.$$

Let the diameter be parallel to the line (1), then the above is

$$\text{A}a_1x + \text{B}\beta_1y + \text{C}z = 0;$$

and the condition that a diameter parallel to (2) may be in this plane, and therefore conjugate to (1), is

$$A\alpha_1\alpha_2 + B\beta_1\beta_2 + C = 0.$$

Similarly,
$$A\alpha_1\alpha_3 + B\beta_1\beta_3 + C = 0,$$
$$A\alpha_2\alpha_3 + B\beta_2\beta_3 + C = 0.$$

These are the conditions that (1) and (3), (2) and (3), shall be conjugate.

Eliminating $\dfrac{A}{C}$, $\dfrac{B}{C}$ between these three equations, we have the property in question, which is therefore generally true, and on the principle of continuity of values will also hold in the case in which the lines (1), (2), (3) are parallel to the principal axes, and therefore mutually perpendicular.

[SECOND SOLUTION.—*Mr. W. Marr, Edinburgh.*]

Let $\left.\begin{array}{l} x=\alpha_1 z+\alpha' \\ y=\beta_1 z+\beta' \end{array}\right\}\ldots(1)$ $\quad \left.\begin{array}{l} x=\alpha_2 z+\alpha'' \\ y=\beta_2 z+\beta'' \end{array}\right\}\ldots(2)$ $\quad \left.\begin{array}{l} x=\alpha_3 z+\alpha''' \\ y=\beta_3 z+\beta''' \end{array}\right\}\ldots\ldots(3)$

be the equations to three lines mutually conjugate to each other in reference to a surface of the second degree.

Since (1) and (2) are conjugate to (3), they must lie in the same or *parallel* planes; let, therefore,

$$Ax + By + z + D = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

be the equation to any one of these planes, then A and B are constant. Because (1) and (2) lie in one of these planes, we must have

$$\left.\begin{array}{l} A\alpha_1 z + B\beta_1 z + z + D' = 0 \\ A\alpha_2 z + B\beta_2 z + z + D' = 0 \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Whence, in consequence of z being indeterminate,

$$\left.\begin{array}{l} A\alpha_1 + B\beta_1 + 1 = 0 \\ A\alpha_2 + B\beta_2 + 1 = 0 \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

By elimination we find $A = \dfrac{\beta_2 - \beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1}$, and $B = \dfrac{\alpha_1 - \alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1} \ldots\ldots(7)$

Now to find the locus of the poles of the planes (4), let (4) be identical with (14) (page 237, *Mathematician,*)

$$\therefore \quad A = \frac{cx_1 + e}{az_1}, \text{ and } B = \frac{by_1}{az_1},$$

and therefore dropping the subscribed numerals, we have for the equation of the locus

$$x = \frac{Aa}{c} z + etc.$$
$$y = \frac{Ba}{b} z + etc.$$

But (3) by supposition is also the locus

$$\therefore \quad \frac{Aa}{c} = \alpha_3 \text{ and } \frac{Ba}{b} = \beta_3 \ldots\ldots\ldots\ldots\ldots(8)$$

Equating the values of A in (7) and (8), we have

$$\frac{\beta_2 - \beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1} = \frac{c}{a}\alpha_3$$

Similarly, $$\frac{\beta_3-\beta_1}{a_1\beta_3-a_3\beta_1}=\frac{c}{a}a_2$$

$$\therefore \frac{\beta_2-\beta_1}{a_1\beta_2-a_2\beta_1}\times\frac{a_1\beta_3-a_3\beta_2}{\beta_3-\beta_1}=\frac{a_3}{a_2}.$$

Multiplying out, transposing, cancelling the common term, and dividing both sides of the resulting equation by $a_1a_2a_3\,\beta_1\beta_2\beta_3$, we obtain the beautiful relation

$$\frac{1}{a_1\beta_3}+\frac{1}{a_2\beta_1}+\frac{1}{a_3\beta_2}=\frac{1}{\beta_1a_3}+\frac{1}{\beta_2a_1}+\frac{1}{\beta_3a_2}.$$

Another proof of this theorem is given by the proposer at page 284.

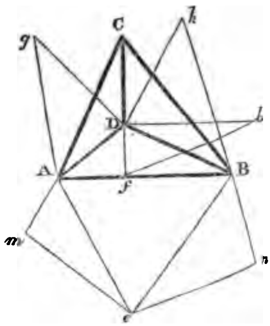
XLI.—*By Pen-and-Ink.*

Give *practicable* constructions of the two following important problems:—

(1.) Given the elevations of two lines (which intersect) above the horizon, and the angle which they form with each other, to find the angle formed by their orthographic projections on the plane of the horizon.

(2.) Given the depressions of two lines below the horizon, and their azimuths, to find the inclination of the plane which contains them to the horizon, and likewise the azimuth of the line of greatest inclination to the horizon, which can be drawn in that plane.

[SOLUTION.—*Mr. Thomas Tate, Training Institution, Battersea.*]



1. Let AC and BC be the two lines, and ADB their orthographic projection.

Conceive the planes ABC, ADC, and BDC to be turned over upon AB as an axis, and laid flat upon the extension of the horizontal plane ADB, the triangle A*me* being the development of the face ADC, the triangle A*e*B that of ACB, and B*en* that of BCD; then $\sphericalangle$ A*e*B = the angle formed by the given lines, $\sphericalangle$ *m*A*e* = the elevation of AC, and $\sphericalangle$ *n*B*e* = the elevation of BC: hence we have the following construction. Draw *em* and *e*A making the included angle equal to the complement of the angle of elevation of the line AC; from any point *m* draw *m*A perpendicular to *em*, intersecting the line *e*A in the point A; draw *e*B making the angle A*e*B equal to the angle formed by the given lines; make the angle B*en* equal to the complement of the angle of elevation of the other side BC, and take *en* equal to *em*, and draw *n*B perpendicular to *en*, intersecting *e*B in the point B; join the points A and B, and construct the triangle ADB, making AD equal to A*m*, and BD equal to B*n*: then the $\sphericalangle$ ADB will be the angle formed by the orthographic projection of the lines.

2. Conceive now the face CDB to be turned over upon DB as an axis, and laid upon the horizontal plane, forming the right-angled triangle BD*h*; and similarly let AD*g* be the development of the face ADC; then $\sphericalangle$ D*h*B and $\sphericalangle$ A*g*D will be respectively the complements of the angles of depression of the lines BC and AC. Hence we have the following construction. Draw *g*D and *g*A making the included angle equal to the complement of the angle of depression of the line AC; from any point D draw DA perpendicular to

Dg, intersecting the line Ag in A; draw DB making the $\angle$ ADB equal to the difference of the angles of azimuth of the two given lines; draw Dh perpendicular to DB, take Dh equal to Dg, and draw hB making the angle DhB equal to the complement of the angle of depression of the line CB; join the points A and B, and draw Df perpendicular to AB; draw Db perpendicular to Df, take Db equal to Dg or Dh, and join the points f and b; then the angle Dfb will be the inclination of the plane which contains the lines, and the position of Df will give the azimuth of the line of greatest inclination.

Mr. Hearn's solution was, in principle, identical with this, and the proposer's solution by the strict methods of Descriptive Geometry is only omitted for want of room. It will, however, appear in Mr. Davies's work on that subject, now preparing for publication.

XLII.—*Mr. Fenwick.*

If the opposite faces of a hexahedron inscribed in a surface of the second order be produced to meet, the three lines of intersection will be in the same plane: Also, if an octahedron circumscribe the same surface (the angular points of the former being the points of contact of the latter), the three lines which join its opposite angular points, two and two, will pass through the same point.

[SOLUTION.—*Mr. G. W. Hearn, R. M. College, Sandhurst.*]

Since the hexahedron is to have eight angular points, its several sides will be plane quadrilaterals. Moreover an infinite number of surfaces of the second order may be made to pass through eight points. Hence any such hexahedron may be considered as described in a surface of the second order. If then the property that when the opposite planes are produced to meet, the three lines of intersection are in the same plane, be *generally* true, it must hold with respect to any hexahedron whose sides are plane quadrilaterals. In order that three lines should be in the same plane they must form a triangle (generally) and therefore intersect two and two. Hence taking two pairs of opposite sides of the hexahedron, it is evident that their four planes must meet in a point which is by no means necessarily true.

To construct a hexahedron having the enunciated property, assume three points in space, and let two planes pass through every pair of the three points; there will thus be six planes, and the plane in which the lines of intersection of opposite sides of the hexahedron formed by them lie, will evidently be that passing through the three assumed points.

Assuming then that a hexahedron thus constructed is inscribed in a surface of the second order, let tangent planes constituting a octahedron pass through the eight angles of the hexahedron, the remaining part of the proposition may be established on well known theorems;—thus

Referring to the construction of the hexahedron, let enveloping cones to the curve surface be drawn from the three points to which the opposite pairs of sides of the hexahedron converge and the three planes of these cones will necessarily intersect in a point.

Again, from the vertices of the octahedron let enveloping cones be drawn. Take two opposite cones—the planes of contact of these cones will be two of the opposite faces of the hexahedron, and the line joining the vertices of the cones will be in two of the three planes mentioned in the last paragraph, and will therefore be the line of their intersection. Hence the three lines

joining opposite vertices will be the three intersections of three planes, and will therefore pass through the same point.

The proposer adds the following

Scholium.—In the investigation (p. 282) that I have given of this theorem I have inferred that it holds generally. This arose from an oversight in respect of the coefficients d, e, f; the conditions which result from the indeterminateness of these not having been fully investigated.

Mr. William Marr, of Edinburgh, also sent a solution.

MATHEMATICAL EXERCISES—*(continued.)*

43.—*By γ.*

If about each of the four triangles into which any quadrilateral figure is divided by its two diagonals a circle be described, the centres of these circles are the four angular points of a parallelogram.

44.—*James Lockhart, Esq., Thistle Grove, Chelsea.*

If x_1 be one of the roots of the cubic equation $x^3 - bx - c = 0$; then the two other roots can be found from either of the following fractional expressions:

$$-\frac{\{4{\cdot}5c + (b^3 - 6{\cdot}75c^2)^{\frac{1}{2}}\}x_1 + b^2}{3bx_1 + 4{\cdot}5c - (b^3 - 6{\cdot}75c^2)^{\frac{1}{2}}}, \text{ or } -\frac{\{4{\cdot}5c - (b^3 - 6{\cdot}75c^2)^{\frac{1}{2}}\}x_1 + b^2}{3bx_1 + 4{\cdot}5c + (b^3 - 6{\cdot}75c^2)^{\frac{1}{2}}}.$$

Required an investigation.

45.—*Mr. Rutherford.*

If in each side of a triangle two points be taken (either all in the sides or all in the sides produced) at equal distances from the angular points; these six points will be in the same conic section.

46.—*By the same.*

The moment of inertia of a circle radius a, revolving about an axis through its centre, and making an angle α with its plane, is

$$Mk^2 = \frac{\pi a^4}{4}(1 + \sin^2\alpha).$$

47.—*By ϕ.*

Over a parabola whose axis is vertical, a string passes, at the extremity of which two weights w and w' are suspended. Shew that if y and y' be the ordinates of the points where they rest in equilibrium we shall have

$$\left(\frac{w'}{y}\right)^2 - \left(\frac{w}{y'}\right)^2 = \frac{w^2 - w'^2}{4m^2}.$$

48.—*By Pen-and-Ink, Charlton.*

Let any two planes X, Y intersect in a line, and three points P, Q, R in that line be taken: draw any three lines PP_1, QQ_1', RR_1 in the plane X, and any three PP_2, QQ_2, RR_2 in the plane Y: let PP_1, QQ_1, RR_1 mutually intersect in a_1, b_1, c_1, and PP_2, QQ_2, RR_2, corresponding to the former ones, intersect in a_2, b_2, c_2: then the lines a_1a_2, b_1b_2, c_1c_2 will pass through the same point, but they can never be all three in the same plane, except when the planes X and Y coincide.

49.—*James Cockle, M. A., Middle Temple.*

Shew that, if $v = 3.2^m - 2$, then (see p. 114)$f^3(v)$ may be decomposed into the sum of m cubes; and make use of this result to obtain a similar theorem for $f^4(v)$.

50.—*By* γ.

If in a hexagon described about a conic section, the first, third, and fifth sides be produced till they form a triangle, and in like manner the second, fourth, and sixth sides till they form another triangle; then the lines joining the opposite pairs of the angular points of these triangles intersect upon the three diagonals of the hexagon.

51.—*Mr. Fenwick.*

If any one of the first six coefficients of the equation

$$ax^2 + by^2 + cx^2 + 2dxy + 2exz + 2fyz + 2gz + 2hy + 2kx + 1 = 0,$$

be arbitrary and independent of the remaining eight coefficients, these eight coefficients will be functions of the co-ordinates of the eight points through which all the surfaces thus represented pass.

52.—*Mr. Thomas Tate,* Battersea Training Institution.*

If the formula $V = \frac{n}{P} + m$, express the relation between the volume and pressure of steam, raised from a unit of water, and if U be put for the *work* performed by the expansion of a cubic foot of water, in the form of steam, between the pressures P and p lbs. per square inch, then

$$e^{\frac{U}{144n}} = \frac{P}{p};$$

where e is the base of the Naperian system of logarithms.

53.—*Mr. Robert Finlay, Professor of Mathematics, Manchester New College.*

If three conic sections have only one common secant (real or ideal), any transversal cuts them in six points which are in involution.

54.—*Mr. Geo. W. Hearn, Professor of Mathematics, R. M. College, Sandhurst.*

The greatest ellipsoid which can be inscribed in a given tetrahedron has its axes coincident with the *mechanical* principal axes (through the centre of gravity) of the tetrahedron.

55.—*By the same gentleman.*

The sum of the squares of the axes of the maximum ellipses inscribed on the faces of any tetrahedron is to the sum of the squares of the axes of its inscribed maximum ellipsoid as thirty-two to nine.

56.—*Mr. Rutherford.*

The upper extremity A of a rod AB, originally in a vertical position, is projected with a given velocity along a smooth groove inclined to the horizon at a given angle: determine the motion of the rod, and the time when the lower extremity B passes the horizontal line through the point of projection.

* We recommend to the attention of our friends, Mr. Tate's elegant little work, recently published,—"A Treatise on Factorial Analysis, with the Summation of Series."

COLLECTION OF MISCELLANEOUS EXERCISES.

*** Under this head we propose to give a series of exercises, without solutions, in various branches of the mathematics, for the use of seminaries and private students.

1. If the perpendiculars let fall from the extremities of the base of a triangle upon the opposite sides be equal to one another the triangle is isosceles.

2. In AB, one of the equal sides of an isosceles triangle ABC, take any point P and produce the other equal side AC till CE = BP, then is the line which joins P and E bisected by the base BC.

3. From one extremity of the base of an isosceles triangle draw a straight line to the opposite side or the opposite side produced making it equal to one of the equal sides, then is the outward angle which this line makes with the base equal to three times one of the equal angles of the isosceles triangle. Does the theorem hold when the triangle is a right-angled one?

4. In a right-angled isosceles triangle let a square be inscribed so that one of its sides may coincide with the hypothenuse, then the square described on the hypothenuse is equal to nine times, and the square on one of the equal sides, to eight times, the inscribed square.

5. If in an equilateral triangle points be taken at equal distances from the angular points, and these points be joined, the triangle thus formed will also be equilateral. Can the points taken at equal distances from the angular points be taken also on the sides produced of the original triangle?

6. If the sides of an equilateral triangle be divided into three equal parts, and the points of division which lie nearest to one and the same angular point be joined two and two, the figure thus formed will be a regular hexagon. Determine also the ratio of the hexagon thus formed to the original triangle.

7. The side of a square inscribed in an equilateral triangle is equal to the excess of four times the perpendicular height of the triangle above its perimeter.

8. If from the angular points of an equilateral triangle ABC equal parts AF, BD, CE be cut off from the angular points, and perpendiculars from the points D, E, F, be let fall on the sides AB, BC, and AC, respectively; then these perpendiculars by their intersections will form an equilateral triangle.

9. In BC, CA, AB sides of a plane triangle ABC take the points D, E, F such that the parts BD, CE and AF are one-third of BC, CA, AB, respectively, and join AD, BE, CF intersecting one another in G, H and K, (G being the point of intersection of AD, FC; and H that of AD, BE;) then are the triangles BDH, CEK, AFG together equal to the middle triangle GHK.

10. If the hypothenuse of a right-angled triangle be produced both ways a distance equal to its own length, and the outward extremities of these be joined to the angular point of the right angle, the sum of the squares of these latter lines is equal to seven times the square of the hypothenuse.

11. If two triangles are upon the same base and their vertices be joined by a line which meets the base or the base produced, the parts of this line between the vertices of the triangles and the base are in the same ratio to *each other* as the areas of the triangles.

12. From the extremity of the base of a triangle draw a line (on the same side as the triangle) perpendicular and equal to the base, join the extremity of this perpendicular and the point where a perpendicular from the vertical angle meets the base; then the point where this joining line meets a second side of the triangle is an angular point of the inscribed square.

13. Produce AB, the base of a triangle ABC, till BE, the produced part, is equal to CD, the perpendicular, then having joined EC, draw BG parallel to EC meeting AC in G, G is an angular point of the inscribed square.

14. In Euclid I., prop. 41, draw any straight line between the parallels AE, BC; then the parts of this line intercepted between the sides of the triangles are equal.

15. If upon the hypothenuse of a right-angled triangle parts be cut off from each extremity equal to the other sides respectively, and from the points of section lines be drawn to the right angle; the triangle is divided into three parts, of which the middle part is equal to the sum of the other two.

16. Draw the diagonal of a square and upon it from one of the angular points take a distance equal to the side of the square; then if from this point a perpendicular be drawn to the diagonal, the part of the perpendicular between the diagonal and a side of the square will be equal to the excess of the diagonal above the side.

17. If the interior angles of a rectangle be bisected by straight lines drawn till they meet, the square thus formed is equal to one half of the magnitude of a square whose side is equal to the difference of the sides of the given rectangle.

18. If the exterior angles of a rectangle be bisected by straight lines drawn as in the last, the square thus formed is equal to one half of the magnitude of a square whose side is equal to the sum of the adjacent sides of the given rectangle.

19. If a rectangle be inscribed in a square, the sum of all the sides of the rectangle is equal to the sum of the two diagonals of the square.

20. The four points of intersection of the four lines bisecting the interior angles of any quadrilateral are in the circumference of a circle.

21. If squares be described upon each side of a quadrilateral, and the angular points of these squares be joined two and two, thus forming four triangles outwardly, then the sum of two opposite triangles will be equal to the sum of the other two opposite triangles.

22. In Euclid III., prop. 31, produce AD till it meets BC produced in H, then shew that the triangle DCH is similar to the triangle ABH.

23. The locus of the points of intersection of all those tangents which are drawn to a given circle and contain an angle of 60° is the circumference of a circle concentric with the given one.

24. In the figure at the end of the third book of Euclid, join BE meeting DF in G, and join GC, GA; then is the angle AGC bisected by BE.

25. If from the sides BC, CA, AB of a plane triangle ABC parts BD, CE; CF, AG; AH, BK, be cut off each equal to the n^{th} part of the sides BC, CA, AB, respectively, and AD, AE, BF, BG, CH, CK be joined: then

$$AD^2 + BF^2 + CH^2 = AE^2 + BG^2 + CK^2.$$

26. In the figure to prop. 11, of the second book of Euclid, prove that

$$AH^2 - HB^2 = AH.\ HB.$$

27. Prove also in the same figure, that

$$AH : HB :: \sqrt{5} - 1 : 3 - \sqrt{5}.$$

28. With reference to the figure to prop. 11, of the fourth book of Euclid, shew *generally* that if in a regular polygon of an *odd* number of sides, lines be drawn as CA, DA to the opposite angle, the angle at the vertex of the isosceles triangle thus formed is to each of the angles at the base, as

$$\frac{n-1}{2} : 1.$$

29. In reference to the same figure as in the last, shew that the area of the pentagon is equal to

$$\frac{3}{2} \text{ of the radius of the circle multiplied by } \frac{5}{6}\ CE.$$

30. In prop. 22, of the third book of Euclid, call the intersection of the diagonals E; in DE take any point F, join AF and CF, then shall

$$\Delta\, ADF : \Delta\, CDF :: AE : EC.$$

Shew that this will also hold when the point is taken in ED produced.

END OF VOL. I.

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